PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Second Memoir on the Compositions of Numbers

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II. *Second Memoir on the Compositions of Numbers.*

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Preamble.

In a Memoir on the Theory of the Compositions of Numbers, read before the Royal Society, November 24, 1892, and published in the 'Philosophical Transactions' for 1893, I discussed the compositions of multipartite numbers by a graphical method. The generating function produced by the method was of the form

$$\frac{1}{1-\Sigma\alpha_1+(1-\lambda)\,\Sigma\alpha_1\alpha_2-(1-\lambda)^2\,\Sigma\alpha_1\alpha_2\alpha_3+\ldots},$$

a symmetrical function of the quantities α.

The investigation of the present paper leads, in part, to the same generating function which is subjected to a close examination. Moreover, the whole research has to do with the compositions of numbers, and appropriately follows the Memoir of 1893.

The problem under investigation, which was brought to my notice by Professor Simon Newcomb, may be stated as follows:—

A pack of cards of any specification is taken—say that there are p cards marked 1, q cards 2, r cards 3, and so on—and, being shuffled, is dealt out on a table; so long as the cards that appear have numbers that are in descending order of magnitude, they are placed in one pack together—equality of number counting as descending order—but directly the descending order is broken a fresh pack is commenced, and so on until all the cards have been dealt. The result of the deal will be m packs containing, in order, a, b, c, ... cards respectively, where, n being the number of cards in the whole pack,

$$(abc\ldots)$$

is some composition of the number n, the numbers of parts in the composition being m.

We have, then, for discussion—

(1) The number of ways of arranging the cards so as to yield a given composition

$$(abc\ldots);$$

(2) The number of arrangements which lead to a distribution into exactly m packs.

These problems, and many others of a like nature, are solved in this paper.

The first of the two questions has given rise to two new symmetric functions,

$$h_{abc\ldots},\ a_{abc\ldots},$$

of great interest, which supply the complete solution. The second gives rise to the same generating function that presented itself in the first Memoir. It is here attacked by the calculus of symmetric function differential operators, and a number of new results obtained.

If the whole pack be specified by the partition

$$(pqr\ldots),$$

there is a one-to-one correspondence between the arrangements which lead to a distribution into m packs and the principal compositions, involving $m-1$ essential nodes, of the multipartite number

$$(\overline{pqr\ldots}).$$

Part I. is concerned with an elementary theory of the case in which the cards are all numbered differently.

The general case, which is more difficult, is dealt with in Part II.

To make what follows clear to the reader, I commence with some elementary notions concerning the connection between the partitions and compositions of numbers on the one hand, and permutations and combinations of things on the other hand, and I also specify and describe the nomenclature and notation that I have found it convenient to adopt. A suitable notation is, indeed, of the first importance in this subject, as I hope to make evident as the investigation proceeds.

INTRODUCTORY.

Art. 1. Any succession of numbers, written down from left to right at random, such as

142771,

is termed a "composition" of the number which is the sum of the numbers.

If the numbers be arranged in descending order from left to right,

774211,

the succession is termed a "descending partition," or simply a "partition" of the number which is the sum of the numbers.

Or, if we arrange in ascending order of magnitude,

112477,

the succession may be termed an "ascending partition."

Generally, in speaking of partitions, we understand that the descending order is meant; but it is convenient sometimes to consider them as being defined by an ascending order.

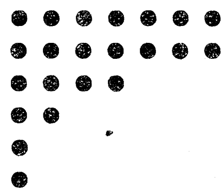
There is no other method of ordering a collection of numbers which is of general application.

We see that the same collection of numbers gives rise to only one partition, but, by permutation, to more than one composition.

Art. 2. Both partitions and compositions have an appropriate graphical representation. That of a partition was first given by FERRERS, and the notion was elaborated by SYLVESTER during the time he was at the Johns Hopkins University in Baltimore, U.S.A. It consisted merely in writing a row of nodes, or units, corresponding to each number (or part) of the partition, the left-hand nodes of the rows being placed in a vertical line. Thus

774211

is denoted by



Art. 3. A trial will show that this method is not suited to compositions. One method, effective for certain purposes, was given by the author.* To indicate it, consider the composition

142

of the number 7.

·—∗—·—·—·—∗—·—·

We take seven segments on a line, and place nodes, ∗, so that the line is divided off into 1, 4 and 2 segments respectively in order. The conjugate composition is reached from this by suppressing the existing nodes and placing nodes at the points of division which are free from nodes.

Thus

·—·—∗—∗—∗—·—∗—·

denotes the composition

21121.

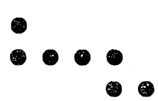
Art. 4. There is a more illuminating mode of representation which is here given, it is believed, for the first time; it is akin to the method of FERRERS, and enables methods of research which SYLVESTER'S exertions have made familiar.

It consists in taking rows of nodes in order and placing the left-hand node of any row vertically beneath the right-hand node of the previous row.

Thus

142

is denoted by

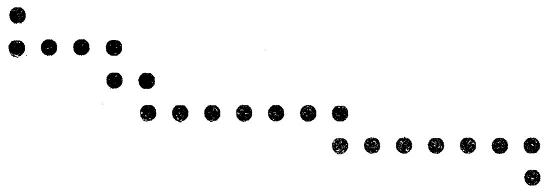


* "Memoir on the Theory of the Compositions of Numbers," 'Phil. Trans. Roy. Soc.,' 1893.

and

142771

by



This graph is read horizontally; the conjugate is obtained by reading vertically, giving

211221111121111112,

or, in brief notation,

$21^2 2^2 1^5 2 1^5 2.$

We may also read the graph horizontally from bottom to top and vertically from right to left, obtaining generally four compositions from the graph.

The graph is a zig-zag one and will be, without doubt, an important instrument of research.

Part I.—Section 1.

Art. 5. Consider the permutations of the first n integers, and for simplicity take $n = 9$.

Writing down a permutation at random,

31 | 4 | 5 | 92 | 76 | 8,

it is clear that lines can be drawn separating the numbers into compartments in such wise that in each compartment the numbers are in descending order of magnitude. We can then write down a succession of numbers which describe the size of the compartments, proceeding from left to right, and thus arrive at a composition

211221

of the number 9.

I say that the permutation under examination has a descending specification

(211221) or $(21^2 2^2 1)$.

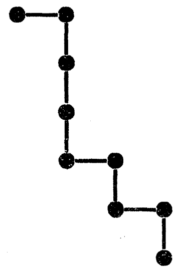
Similarly, from the ascending character

3 | 1459 | 27 | 68

of the same permutation, I say that the ascending specification is

(1422) or (142^2),

where it is to be noticed that 142^2 is the composition of 9 which is conjugate to $21^2 2^2 1$, the composition which specifies the descending character. This is shown by the zig-zag graph



Art. 6. We can now formulate the question: Of the permutations of the first n numbers, how many have a descending specification denoted by a given composition of the number n? Whatever the answer, it is clear that the same answer must, in general, be given for three other compositions, viz., the three others associated with the zig-zag graph. In fact, from

314592768 of specification 211221,

we derive

867295413 ,, 2241;

and from these two by changing the number m into $n-m+1$,

796518342 of specification 1422,

243815679 ,, 122112

and so forth.

In two cases there are two associated compositions instead of four, viz.:—

(i) When the composition reads the same as its inverse (that is the same from left to right as from right to left),

(ii) When the conjugate and the inverse are identical, as in 221, whose conjugate is 122.

*The number of self-inverse compositions of an even number $2m$ and of an uneven number $2m+1$ is

$$2^m.$$

The number of inverse-conjugate compositions of an uneven number $2m+1$ is

$$2^m.$$

Hence, in the present theory, the number of different numbers that appear in the case of an even number $2m$ is, since the whole number of compositions is 2^{2m-1},

$$\tfrac{1}{2}\cdot 2^m+\tfrac{1}{4}\left(2^{2m-1}-2^m\right),$$
$$=2^{m-2}\left(2^{m-1}+1\right);$$

and, in the case of an uneven number $2m+1$,

$$\tfrac{1}{2}2^m+\tfrac{1}{2}2^m+\tfrac{1}{4}\left(2^{2m}-2^{m+1}\right),$$
$$=2^{m-1}\left(2^{m-1}+1\right);$$

viz., it is

$$2^{n-3}+2^{1/2(n-4)},$$

or

$$2^{n-3}+2^{1/2(n-3)},$$

according as n is even or uneven.

* See "Memoir on the Theory of the Compositions of Numbers," 'Phil. Trans. Roy. Soc.,' 1893.

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Art. 7. Let N $(abc\ldots)$ denote the number of permutations of the first n integers which have a descending specification denoted by the composition

$$(abc\ldots)$$

of the number n.

Obviously

$$\mathrm{N}(a) = 1, \quad a = n.$$

To determine N (ab), $a+b = n$, separate the n integers into two groups, a left-hand group of a numbers chosen at random and a right-hand group of the remaining b numbers. This can be done in

$$\binom{n}{a} \text{ different ways.}$$

$\left[\text{I write } \dfrac{n!}{a!\,(n-a)!} = \dbinom{n}{a} \text{ in a common notation}\right]$; now arrange each group of numbers in descending order of magnitude for each of the $\binom{n}{a}$ separations; we thus obtain each of the permutations enumerated by N (a, b) and the one permutation enumerated by N $(a+b)$.

Hence

$$\mathrm{N}(ab) + \mathrm{N}(a+b) = \binom{n}{a},$$

or

$$\mathrm{N}(ab) = \binom{n}{a} - \binom{n}{a+b} = \binom{n}{a} - 1.$$

Again, to find N (abc), we separate the n integers into three groups containing a, b, and c integers respectively; this can be done in

$$\frac{n!}{a!\,b!\,c!}$$

different ways; placing the numbers in each group in descending order, we obtain all the permutations enumerated by

$$\mathrm{N}(abc), \quad \mathrm{N}(a+b,\, c), \quad \mathrm{N}(a,\, b+c), \quad \mathrm{N}(a+b+c).$$

Hence

$$\mathrm{N}(abc) + \mathrm{N}(a+b,\, c) + \mathrm{N}(a,\, b+c) + \mathrm{N}(a+b+c) = \frac{n!}{a!\,b!\,c!};$$

leading to

$$\mathrm{N}(abc) = \frac{n!}{a!\,b!\,c!} - \frac{n!}{(a+b)!\,c!} - \frac{n!}{a!\,(b+c)!} + \frac{n!}{(a+b+c)!},$$

where $a+b+c = n$.

Similarly we find

$$\mathrm{N}(abcd) = \frac{n!}{a!\,b!\,c!\,d!} - \frac{n!}{(a+b)!\,c!\,d!} - \frac{n!}{a!\,(b+c)!\,d!} - \frac{n!}{a!\,b!\,(c+d)!}$$

$$+ \frac{n!}{(a+b)!\,(c+d)!} + \frac{n!}{(a+b+c)!\,d!} + \frac{n!}{a!\,(b+c+d)!} - \frac{n!}{(a+b+c+d)!},$$

where $a+b+c+d = n$.

The general law is clear; the letters a, b, c, d are always in order in the denominators and the sign of a fraction depends upon the number of factors in its denominator.

We can thus calculate the number of permutations appertaining to each of the 2^{n-1} compositions of n.

It has been established independently, by the aid of the zig-zag graph, that these numbers

$$N(\ldots)$$

are equal in four's or in two's.

Art. 8. The sum of the numbers N (...) is of course $n!$

The details of the above results for

$$n = 2, 3, 4, 5, 6$$

are given for easy reference.

$$\begin{aligned} N(2) &= 1 \qquad 1 \\ N(1^2) &= 1 \qquad 1 \\ & \qquad\quad \overline{2} = 2! \end{aligned}$$

$$\begin{aligned} N(3) = N(1^3) &= 1 \qquad 2 \\ N(21) = N(12) &= 2 \qquad 4 \\ & \qquad\quad \overline{6} = 3! \end{aligned}$$

$$\begin{aligned} N(4) = N(1^4) &= 1 \qquad 2 \\ N(31) = N(13) = N(21^2) = N(1^2 2) &= 3 \qquad 12 \\ N(22) = N(121) &= 5 \qquad 10 \\ & \qquad\quad \overline{24} = 4! \end{aligned}$$

$$\begin{aligned} N(5) = N(1^5) &= 1 \qquad 2 \\ N(41) = N(14) = N(21^3) = N(1^3 2) &= 4 \qquad 16 \\ N(32) = N(23) = N(121^2) = N(1^2 21) &= 9 \qquad 36 \\ N(31^2) = N(1^2 3) &= 6 \qquad 12 \\ N(2^2 1) = N(12^2) &= 16 \qquad 32 \\ N(131) = N(212) &= 11 \qquad 22 \\ & \qquad\quad \overline{120} = 5! \end{aligned}$$

		$N(6)$	$= N(1^6)$	$= 1$	2
$N(51)$	$= N(15)$	$= N(21^4)$	$= N(1^42)$	$= 5$	20
$N(42)$	$= N(24)$	$= N(1^321)$	$= N(121^3)$	$= 14$	56
		$N(3^2)$	$= N(1^221^2)$	$= 19$	38
$N(41^2)$	$= N(1^24)$	$= N(31^3)$	$= N(1^33)$	$= 10$	40
		$N(141)$	$= N(21^22)$	$= 19$	38
$N(321)$	$= N(123)$	$= N(2^21^2)$	$= N(1^22^2)$	$= 35$	140
$N(312)$	$= N(213)$	$= N(1^231)$	$= N(131^2)$	$= 26$	104
$N(132)$	$= N(231)$	$= N(2121)$	$= N(1212)$	$= 40$	160
		$N(2^3)$	$= N(12^21)$	$= 61$	122
					$720 = 6!$

Art. 9. Some simple summations are obtainable from elementary considerations.

In regard to the permutations of the first n integers, let

$$\Sigma N(s\ldots),$$

where $s<n$, denote the sum of all numbers N (...), such that s is the first number in the specifying composition. Take any $s+1$ of the numbers

$$1, 2, 3, \ldots n,$$

and arrange them from left to right in such wise that the first s numbers are in descending order and the $s+1^{\text{th}}$ number greater than the s^{th}; this can be done in

$$s\binom{n}{s+1} \text{ ways};$$

the remaining $n-s-1$ numbers can be arranged in $(n-s-1)!$ ways, so that, placing them to the right of the former, we arrive at the result

$$\Sigma N(s\ldots) = s\frac{n!}{(s+1)!}.$$

Art. 10. Again, denoting by

$$\Sigma N(1^{s-1}\ldots)$$

the sum of all numbers N (...) of which the specifying compositions commence with exactly $s-1$ units, the consideration of the properties of conjugate zig-zag graphs establishes that

$$\Sigma N(1^{s-1}\ldots) = \Sigma N(s\ldots),$$

with a single exception where $s=n$; *e.g.*,

$$\Sigma N(1^0\ldots) = {}^*\Sigma N(1\ldots) = \frac{n!}{2!},$$

$$^\dagger\Sigma N(1\ldots) = \Sigma N(2\ldots) = 2\frac{n!}{3!},$$

and so on.

* No restriction is placed upon the number next to the unit in this case.

† Here the number following the unit must be >1.

Art. 11. Again, for the summation

$$\Sigma N(1^s\ldots),$$

where the composition begins with *at least* s units, we easily obtain the value

$$\frac{n!}{(s+1)!}.$$

The Multiplication Theorem.

Art. 12. A fundamental property of the numbers $N(\ldots)$ will be established from elementary considerations; it will, later on in the paper, be generalised.

Let

$$N(\alpha_1\alpha_2\ldots\alpha_s)$$

be derived from the permutations of p different integers, and

$$N(\alpha_{s+1}\alpha_{s+2}\ldots\alpha_{s+t})$$

from the permutations of $n-p$ different integers; it is to be shown that

$$\binom{n}{\alpha_1+\alpha_2+\ldots+\alpha_s} N(\alpha_1\alpha_2\ldots\alpha_s)\, N(\alpha_{s+1}\alpha_{s+2}\ldots\alpha_{s+t})$$
$$= N(\alpha_1\alpha_2\ldots\alpha_{s+t}) + N(\alpha_1\alpha_2\ldots\alpha_{s-1},\ \alpha_s+\alpha_{s+1},\ \alpha_{s+2}\ldots\alpha_{s+t}),$$

where on the right the reference is to the permutations of n different integers.

Out of the n numbers

$$1,\ 2,\ 3,\ldots n,$$

we can select

$$\alpha_1+\alpha_2+\ldots+\alpha_s$$

numbers in

$$\binom{n}{\alpha_1+\alpha_2+\ldots+\alpha_s}$$

ways, and arrange each selection, so as to have a descending specification

$$(\alpha_1\alpha_2\ldots\alpha_s),$$

in

$$N(\alpha_1\alpha_2\ldots\alpha_s) \text{ ways};$$

the remaining numbers can be arranged, to have a descending specification

$$(\alpha_{s+1}\alpha_{s+2}\ldots\alpha_{s+t}),$$

in

$$N(\alpha_{s+1}\alpha_{s+2}\ldots\alpha_{s+t}) \text{ ways};$$

placing the latter to the right of the former there appears

$$\binom{n}{\alpha_1+\alpha_2+\ldots+\alpha_s} N(\alpha_1\alpha_2\ldots\alpha_s)\, N(\alpha_{s+1}\alpha_{s+2}\ldots\alpha_{s+t})$$

arrangements.

Now, combining the two sets of numbers, we find that either there is or there is not a break in the descending order between

$$\alpha_s \text{ and } \alpha_{s+1};$$

hence the number of arrangements is also

$$N(\alpha_1\alpha_2\ldots\alpha_{s+t}) + N(\alpha_1\alpha_2\ldots\alpha_{s-1},\ \alpha_s+\alpha_{s+1},\ \alpha_{s+2}\ldots\alpha_{s+t}). \quad \text{Q.E.D.}$$

Art. 13. Regarded as a numerical theorem, the multiplication is commutative, but in regard to form it is not commutative; thus, by considering the multiplication

$$N(\alpha_{s+1}\alpha_{s+2}\ldots\alpha_{s+t})\, N(\alpha_1\alpha_2\ldots\alpha_s),$$

we obtain the linear relation

$$N(\alpha_1\alpha_2\ldots\alpha_{s+t}) + N(\alpha_1\alpha_2\ldots\alpha_{s-1},\ \alpha_s+\alpha_{s+1},\ \alpha_{s+2}\ldots\alpha_{s+t})$$
$$= N(\alpha_{s+1}\alpha_{s+2}\ldots\alpha_{s+t}\alpha_1\ldots\alpha_s) + N(\alpha_{s+1}\ldots\alpha_{s+t-1},\ \alpha_{s+t}+\alpha_1,\ \alpha_2\ldots\alpha_s).$$

Observe also that the order of the numbers in brackets in any number N (...) can be reversed at pleasure and thus new forms of results obtained.

As a verification: from the tables

$$\binom{5}{3} N(12)\, N(11) = N(121^2) + N(131) = N(1^3 2) + N(12^2);$$
$$10 \;.\; 2 \;.\; 1 = 9 + 11 = 4 + 16$$

$$N(123) + N(15) = N(312) + N(42).$$
$$35 + 5 = 26 + 14$$

The fact that the multiplication is not commutative formally is of great importance in the theory of these numbers.

Art. 14. Extending the theorem to the product of three numbers

$$N(\alpha_1\alpha_2\ldots\alpha_s), \quad N(b_1b_2\ldots b_t), \quad N(c_1c_2\ldots c_u),$$

we find

$$\frac{n!}{(\Sigma\alpha)!\,(\Sigma b)!\,(\Sigma c)!} N(\alpha_1\alpha_2\ldots\alpha_s)\, N(b_1b_2\ldots b_t)\, N(c_1c_2\ldots c_u)$$
$$= N(\alpha_1\ldots\alpha_s b_1\ldots b_t c_1\ldots c_u) + N(\alpha_1\ldots\alpha_{s-1},\ \alpha_s+b_1,\ b_2\ldots b_t c_1\ldots c_u)$$
$$+ N(\alpha_1\ldots\alpha_s b_1\ldots b_{t-1},\ b_t+c_1,\ c_2\ldots c_u) + N(\alpha_1\ldots\alpha_{s-1},\ \alpha_s+b_1,\ b_2\ldots b_{t-1},\ b_t+c_1,\ c_2\ldots c_u).$$

We may, in general, give the right-hand side 3! different forms corresponding to the 3! permutations of the numbers N (...) on the sinister.

If we take the product of m numbers N (...), to form the dexter, we combine the last integer of a number N (...) with the first integer of the next following number N (...),

$$
\begin{array}{llll}
0 & \text{times in} & 1 & \text{way}, \\
1 & \quad ,, & \binom{m-1}{1} & \text{ways}, \\
2 & \quad ,, & \binom{m-1}{2} & \quad ,, \\
\cdot & \quad ,, & \cdot & \quad ,, \\
m-1 & \quad ,, & \binom{m-1}{m-1} & \quad ,, ;
\end{array}
$$

hence 2^{m-1} numbers N (...) present themselves on the dexter.

Not counting reversals of order, the dexter can, in general, be given as many different forms as there are permutations of the numbers N (...) on the sinister. Counting reversals, the number of different forms is further multiplied by 2^m, subject to a diminution when one or more of the numbers N (...) is self-inverse.

Applications of the Theorem.

Art. 15. The theorems, already arrived at above, are particular cases of multiplication. Thus the formulæ, of which

$$\mathrm{N}(abc) + \mathrm{N}(a+b, c) + \mathrm{N}(a, b+c) + \mathrm{N}(a+b+c) = \frac{n!}{a!\,b!\,c!}$$

is a type, are equivalent to results, of which

$$\frac{n!}{a!\,b!\,c!}\mathrm{N}(a)\,\mathrm{N}(b)\,\mathrm{N}(c) = \mathrm{N}(abc) + \mathrm{N}(a+b, c) + \mathrm{N}(a, b+c) + \mathrm{N}(a+b+c)$$

is representative, since $\mathrm{N}(a) = \mathrm{N}(b) = \mathrm{N}(c) = 1.$

That the sum of all numbers N (...), of given weight n, is $n!$ is shown by the formula

$$n!\{\mathrm{N}(1)\}^n = \underset{\cdots}{\Sigma}\mathrm{N}(\ldots);$$

since on the dexter occurs an N (...) corresponding to every composition of n.

Art. 16. Suppose that it is required to find the sum of all numbers N (...), of given weight, which are such that each associated composition commences with a given series of numbers

$$a_1 a_2 \ldots a_m,$$

or, in other words, suppose we wish to make the summation indicated by

$$\underset{\cdots}{\Sigma}\mathrm{N}(a_1 a_2 \ldots a_m \ldots);$$

the solution is given at once by

$$\frac{n!}{(\Sigma a+1)!}\mathrm{N}(a_1 a_2 \ldots a_m 1)\{\mathrm{N}(1)\}^{n-\Sigma a-1} = \underset{\cdots}{\Sigma}\mathrm{N}(a_1 a_2 \ldots a_m \ldots);$$

for, by the multiplication process, the unit which terminates $\mathrm{N}(a_1 a_2 \ldots a_m 1)$, combined with

$$\{\mathrm{N}(1)\}^{n-\Sigma a-1},$$

gives every composition of the number

$$n-\Sigma a.$$

Hence, since $\mathrm{N}(1) = 1$,

$$\underset{\cdots}{\Sigma}\mathrm{N}(a_1a_2\ldots a_m\ldots) = \frac{n!}{(\Sigma a+1)!}\mathrm{N}(a_1a_2\ldots a_m1).$$

Art. 17. By varying the order of the factors, on the sinister of the multiplication formula, a variety of interesting results present themselves; thus

$$\frac{n!}{(\Sigma a+1)!}\{\mathrm{N}(1)\}^p\,\mathrm{N}(a_1a_2\ldots a_m1)\{\mathrm{N}(1)\}^{n-\Sigma a-p-1} = \Sigma\Sigma\mathrm{N}(\ldots a'_1a_2a_3\ldots a_m\ldots);$$

where after a_m, on the dexter, occurs every composition of

$$n-\Sigma a-p;$$

and the portion $\ldots a'_1$

includes every composition of $p+a_1$

which terminates with a number not less than a_1.

Hence, for such a summation,

$$\Sigma\Sigma\mathrm{N}(\ldots a'_1a_2\ldots a_m\ldots) = \frac{n!}{(\Sigma a+1)!}\mathrm{N}(a_1a_2\ldots a_m1);$$

a formula which is independent of p.

Art. 18. In particular from

$$\{\mathrm{N}(1)\}^{n-\Sigma a-1}\,\mathrm{N}(a_1a_2\ldots a_m1)$$

we obtain

$$\Sigma\mathrm{N}(\ldots a'_1a_2\ldots a_m1) = \frac{n!}{(\Sigma a+1)!}\mathrm{N}(a_1a_2\ldots a_m1);$$

wherein the summation is for every composition of

$$n-a_2-\ldots-a_m-1$$

which terminates with a number not less than a_1.

E.g., for $n=6$, $a_1=1$, $a_2=1$,

$$\mathrm{N}(41^2)+\mathrm{N}(131^2)+\mathrm{N}(2^21^2)+\mathrm{N}(1^221^2) = \frac{6!}{4!}\mathrm{N}(211).$$

$$10 + 26 + 35 + 19 = 6.5.3$$

Art. 19. As another example of the power of the theorem, let

$$\Sigma\mathrm{N}(a_1a_2\ldots a_{m_1}\ldots b_1b_2\ldots b_{m_2})$$

(the numbers a_1, $a_2\ldots a_{m_1}$, b_1, $b_2\ldots b_{m_2}$ being given) denote a summation in regard to compositions of

$$n-\Sigma a-\Sigma b$$

placed between a_{m_1} and b_1; we obtain

$$\underset{\cdots}{\Sigma}\mathrm{N}\,(a_1a_2\ldots a_{m_1}\ldots b_1b_2\ldots b_{m_2})$$

$$= \frac{n!}{(\Sigma a+1)!\,(\Sigma b+1)!}\mathrm{N}\,(a_1a_2\ldots a_{m_1}1)\,\{\mathrm{N}\,(1)\}^{n-\Sigma a-\Sigma b-2}\,\mathrm{N}\,(1b_1b_2\ldots b_{m_2}),$$

$$= \frac{n!}{(\Sigma a+1)!\,(\Sigma b+1)!}\mathrm{N}\,(a_1a_2\ldots a_{m_1}1)\,\mathrm{N}\,(1b_1b_2\ldots b_{m_2}),$$

$$= \frac{n!}{(\Sigma a+\Sigma b+2)!}\{\mathrm{N}\,(a_1a_2\ldots a_{m_1}1^2b_1b_2\ldots b_{m_2})+\mathrm{N}\,(a_1a_2\ldots a_{m_1}2b_1b_2\ldots b_{m_2})\}.$$

By varying the order of the factors, other summations, leading to the same numerical result, can be effected.

Art. 20. Consider next the multiplication

$$\frac{n!}{(s_1+2)!\,(s_2+2)!\,(s_3+2)!}$$
$$\times\{\mathrm{N}\,(1)\}^{w_1-1}\,\mathrm{N}\,(1^{s_1+2})\,\{\mathrm{N}\,(1)\}^{w_2-2}\,\mathrm{N}\,(1^{s_2+2})\,\{\mathrm{N}\,(1)\}^{w_3-2}\,\mathrm{N}\,(1^{s_3+2})\,\{\mathrm{N}\,(1)\}^{w_4-1};$$

wherein, $\Sigma w+\Sigma s = n$,

w_1, w_2, w_3, w_4 are numbers not less than unity,

s_1, s_2, s_3 are any numbers, zero not excluded.

The result of the multiplication consists of numbers N (...), such that there is

(i) A composition of w_1 followed by s_1 units, succeeded by
(ii) A composition of w_2 followed by s_2 units, succeeded by
(iii) A composition of w_3 followed by s_3 units, succeeded by
(iv) A composition of w_4;

and the dexter is the sum of all such numbers N (...).

Denoting this sum by

$$\Sigma\mathrm{N}\,(w_1 1^{s_1} w_2 1^{s_2} w_3 1^{s_3} w_4),$$

we find that its value is

$$\frac{n!}{(s_1+2)!\,(s_2+2)!\,(s_3+2)!},$$

since each number N (...) occurring in the product on the sinister has unity for its value.

Hence, in general, the remarkable theorem,

$$\Sigma\mathrm{N}\,(w_1 1^{s_1} w_2 1^{s_2} w_3 1^{s_3}\ldots) = \frac{n!}{(s_1+2)!(s_2+2)!(s_3+2)!\ldots};$$

showing that the sum depends merely upon the numbers

$$s_1,\ s_2,\ s_3,\ \ldots,$$

and not at all upon the numbers

$$w_1,\ w_2,\ w_3,\ \ldots .$$

Observe that w_1 and the final number of the composition may or may not be unity, and that every composition of n may be written in the form

$$w_1 1^{s_1} w_2 1^{s_2} w_3 1^{s_3} \ldots .$$

If

$$s_1 = s_2 = s_3 = \ldots = 0,$$

$$\Sigma \mathrm{N}\,(w_1 w_2 w_3 w_4 \ldots) = \frac{n!}{2!\,2!\,2!\ldots};$$

and, in particular,

$$\Sigma \mathrm{N}\,(w_1 w_2 \ldots w_m) = \frac{n!}{(2!)^{m-1}},$$

wherein $w_2,\ w_3,\ \ldots\ w_{m-1}$ are non-unitary, but w_1, w_m may or may not be unitary.

As a simple example take

$$w_1 = 1, \quad s_1 = 4, \quad w_2 = 1,$$

so that

$$\mathrm{N}\,(1^6) = \Sigma \mathrm{N}\,(1,\ 1^4,\ 1) = \frac{6!}{(4+2)!} = 1,$$

a verification.

Art. 21. A more general theorem is yielded by

$$\frac{n!}{(\Sigma p+2)!\,(\Sigma q+2)!\,(\Sigma r+2)!}$$

$$\times \{\mathrm{N}(1)\}^{w_1-1}\mathrm{N}\,(1p_1\ldots p_{m_1}1)\,\{\mathrm{N}(1)\}^{w_2-2}\mathrm{N}\,(1q_1\ldots q_{m_2}1)\,\{\mathrm{N}(1)\}^{w_3-2}\mathrm{N}\,(1r_1\ldots r_{m_3}1)\,\{\mathrm{N}(1)\}^{w_4-1},$$

$$= \Sigma \mathrm{N}\,(w_1 p_1 \ldots p_{m_1} w_2 q_1 \ldots q_{m_2} w_3 r_1 \ldots r_{m_3} w_4),$$

wherein

$$p_1 \ldots p_{m_1},$$
$$q_1 \ldots q_{m_2},$$
$$r_1 \ldots r_{m_3},$$

are given integers and the summation indicated on the dexter is in respect of the whole of the compositions of the numbers

$$w_1, \quad w_2, \quad w_3, \quad w_4,$$

where

$$0 \eqslantless w_1 - 1, \quad w_2 - 2, \quad w_3 - 2 \quad \text{and} \quad w_4 - 1.$$

The value of the sum is thus

$$\frac{n!}{(\Sigma p+2)!\,(\Sigma q+2)!\,(\Sigma r+2)!}\,\mathrm{N}\,(1p_1\ldots p_{m_1}1)\,\mathrm{N}\,(1q_1\ldots q_{m_2}1)\,\mathrm{N}\,(1r_1\ldots r_{m_3}1),$$

which, by the multiplication theorem, may be given the form

$$\frac{n!}{(\Sigma p+\Sigma q+\Sigma r+6)!}\times[\ \mathrm{N}\,(1p_1\ldots p_{m_1}1^2q_1\ldots q_{m_2}1^2r_1\ldots r_{m_3}1)$$
$$+\mathrm{N}\,(1p_1\ldots p_{m_1}2q_1\ldots q_{m_2}1^2r_1\ldots r_{m_3}1)$$
$$+\mathrm{N}\,(1p_1\ldots p_{m_1}1^2q_1\ldots q_{m_2}2r_1\ldots r_{m_3}1)$$
$$+\mathrm{N}\,(1p_1\ldots p_{m_1}2q_1\ldots q_{m_2}2r_1\ldots r_{m_3}1)].$$

Evidently, from the above, comprehensive results can be obtained from the multiplication theorem.

Section 2.

Art. 22. The next problem I propose to solve is that of determining the number of the permutations of the first n integers, whose descending specifications contain a given number of integers, or, in other words, whose associated compositions involve a given number of parts. The solution is implicitly contained in a paper I wrote in the year 1888.*

Let N_m denote the number of permutations associated with compositions containing exactly m parts.

In the paper quoted, I had under view a collection of objects of any species—say p of one sort, q of a second sort, r of a third, and so on—and defined the objects as to species by these numbers placed in brackets. I thus formed a partition

$$(pqr\ldots)$$

of the number n, such partition being the species definition of the objects.

As equalities may occur between the numbers p, q, r, …, I took, as a more general definition, the partition

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots),$$

where $\Sigma\pi p = n$.

In the case under consideration, where the integers (or objects) are all different, the species definition is the partition

$$(1^n).$$

I proved, in the general case, that the number of ways of distributing the objects, into m different parcels, is given by the series

$$\begin{aligned}\mathrm{F}_m = &\binom{m+p_1-1}{p_1}^{\pi_1}\binom{m+p_2-1}{p_2}^{\pi_2}\binom{m+p_3-1}{p_3}^{\pi_3}\ldots\\ &-\binom{m}{1}\binom{m+p_1-2}{p_1}^{\pi_1}\binom{m+p_2-2}{p_2}^{\pi_2}\binom{m+p_3-2}{p_3}^{\pi_3}\ldots\\ &+\binom{m}{2}\binom{m+p_1-3}{p_1}^{\pi_1}\binom{m+p_2-3}{p_2}^{\pi_2}\binom{m+p_3-3}{p_3}^{\pi_3}\ldots\\ &-\ldots.\end{aligned}$$

* "Symmetric Functions and the Theory of Distributions," 'Proc. L. M. S.,' vol. xix., p. 226.

For the case in hand, $p_1 = 1$, $\pi_1 = n$,

$$F_m = m^n - \binom{m}{1}(m-1)^n + \binom{m}{2}(m-2)^n - \binom{m}{3}(m-3)^n + \ldots .$$

Art. 23. I shall prove that

$$N_m = m^n - \binom{n+1}{1}(m-1)^n + \binom{n+1}{2}(m-2)^n - \binom{n+1}{3}(m-3)^n + \ldots .$$

For consider the arrangements enumerated by F_m. Place the compartments (or parcels) in order, from left to right, in any one such arrangement, and, in each compartment, place the integers in descending order of magnitude. The arrangement is obviously one of those enumerated by

$$N_m,\ N_{m-1},\ N_{m-2},\ \ldots \text{ or } N_1.$$

In the whole of the arrangements, enumerated by F_m, thus treated, each arrangement enumerated by N_m will occur once only.

Let the illustration denote an arrangement enumerated by N_{m-1}. Each segment denotes an integer, and the $m-2$ vertical lines separate the integers into compartments.

By placing an extra vertical line at one of the unoccupied points of division, we obtain an arrangement enumerated by F_m. This can be done in $(n-1)-(m-2)$ different ways, showing that the particular arrangement, enumerated by N_{m-1}, is derivable by obliteration of a vertical line from

$$n-m+1$$

different arrangements enumerated by F_m.

Hence, the forms F_m include the forms N_{m-1} each $n-m+1$ times.

Again, let the illustration denote an arrangement enumerated by N_{m-s}. By placing s extra vertical lines, at unoccupied points of division, we obtain an arrangement enumerated by F_m. This can be done in

$$\binom{n-m+s}{s}$$

different ways; showing that the particular arrangement, enumerated by N_{m-s}, is derivable, by obliteration of s vertical lines, from

$$\binom{n-m+s}{s}$$

different arrangements enumerated by F_m.

Hence the forms F_m include the forms N_{m-s} each

$$\binom{n-m+s}{s} \text{ times.}$$

Hence

$$F_m = N_m + \binom{n-m+1}{1} N_{m-1} + \binom{n-m+2}{2} N_{m-2} + \ldots + \binom{n-1}{m-1} N_1.$$

Thence it is easy to show that

$$N_m = F_m - \binom{n-m+1}{1} F_{m-1} + \binom{n-m+2}{2} F_{m-2} - \ldots + (-)^{m+1} \binom{n-1}{m-1} F_1 ;$$

and also

$$N_m = m^n - \binom{n+1}{1}(m-1)^n + \binom{n+1}{2}(m-2)^n - \ldots + (-)^{m+1}\binom{n+1}{m-1} 1^n.$$

The relation $\sum_1^n N_m = n!$ may be verified.

Art. 24. It follows at once, from the zig-zag graphs, that

$$N_m = N_{n-m+1}.$$

Some of the simplest results are

$n =$	N_1.	N_2.	N_3.	N_4.	N_5.	N_6.
1	1					
2	1	1				
3	1	4	1			
4	1	11	11	1		
5	1	26	66	26	1	
6	1	57	302	302	57	1

Art. 25. There is another interesting series for N_m.

Let

$$(p+s)^n{}_{-1,0}$$

denote the expansion of

$$(p+s)^n$$

when deprived of the term which is linear in p and of the term independent of p; and put

$$P_s = (1+s)^n{}_{-1,0} ;$$

then

$$N_m = P_{m-1} - \frac{m-2}{m-1}\binom{n}{1} P_{m-2} + \frac{m-3}{m-1}\binom{n}{2} P_{m-3} - \ldots + (-)^m \frac{1}{m-1}\binom{n}{m-2} P_1.$$

I prove a general theorem, of which this is a particular case, later on in the paper.

Art. 26. Considering next p different numbers, defined by the partition

$$(1^p),$$

we have, by a previous definition,

$$\sum_a N(\alpha_1\alpha_2\ldots\alpha_m) = N_{m,1^p};$$

where $\alpha_1, \alpha_2, \alpha_3, \ldots$ are each $\not< 1$ and such that

$$\Sigma\alpha = p.$$

I have written

$$N_{m,1^p}$$

instead of N_m, in order to specify the number of objects (or numbers) subjected to permutation.

Art. 27. I shall now prove that

$$\sum_a\sum_{\cdots} N(\alpha_1\alpha_2\ldots\alpha_m\ldots) = \frac{n!}{(p+1)!}(p-m+1)\,N_{m,1^p},$$

where in

$$N(\alpha_1\alpha_2\ldots\alpha_m\ldots)$$

the number of objects subjected to permutation is n, and the summation is in respect of all permutations such that the sum of the first m numbers in the descending specification is equal to p.

For, by Art. 16,

$$\sum_{\cdots} N(\alpha_1\alpha_2\ldots\alpha_m\ldots) = \frac{n!}{(p+1)!}N(\alpha_1\alpha_2\ldots\alpha_m 1);$$

hence

$$\sum_a\sum_{\cdots} N(\alpha_1\alpha_2\ldots\alpha_m\ldots) = \frac{n!}{(p+1)!}\sum_a N(\alpha_1\alpha_2\ldots\alpha_m 1);$$

and, by the multiplication theorem,

$$(p+1)\,N(\alpha_1\alpha_2\ldots\alpha_m)\,N(1) = N(\alpha_1\alpha_2\ldots\alpha_m 1) + N(\alpha_1\alpha_2\ldots\alpha_m+1);$$

so that

$$\sum_a N(\alpha_1\alpha_2\ldots\alpha_m 1) = (p+1)\,N_{m,1^p} - \sum_a N(\alpha_1\alpha_2\ldots\alpha_m+1);$$

and since

$$\sum_a N(\alpha_1\alpha_2\ldots\alpha_m+1) = N_{m,1^{p+1}} - \sum_a N(\alpha_1\alpha_2\ldots\alpha_{m-1}1),$$

$$\sum_a N(\alpha_1\alpha_2\ldots\alpha_m 1) - \sum_a N(\alpha_1\alpha_2\ldots\alpha_{m-1}1) = (p+1)\,N_{m,1^p} - N_{m,1^{p+1}};$$

whence, by summation,

$$\sum_a N(\alpha_1\alpha_2\ldots\alpha_m 1) = (p+1)\sum_1^m N_{m,1^p} - \sum_1^m N_{m,1^{p+1}};$$

but since

$$N_{m,1^p} = m^p - \binom{p+1}{1}(m-1)^p + \binom{p+1}{2}(m-2)^p - \ldots$$

$$\sum_1^m N_{m,1^p} = m^p - \binom{p}{1}(m-1)^p + \binom{p}{2}(m-2)^p - \ldots,$$

so that, substituting,

$$\sum_{a} \mathrm{N}\,(a_1 a_2 \ldots a_m 1) = (p-m+1)\,\mathrm{N}_{m,1^p};$$

hence

$$\sum_{a}\sum_{\cdots} \mathrm{N}\,(a_1 a_2 \ldots a_m \ldots) = \frac{n!}{(p+1)!}(p-m+1)\,\mathrm{N}_{m,1^p}.$$

Art. 28. Further, summing each side with respect to m,

$$\sum_{m}\sum_{a}\sum_{\cdots} \mathrm{N}\,(a_1 a_2 \ldots a_m \ldots)$$

$$= \frac{n!}{(p+1)!}\{p\mathrm{N}_{1,1^p} + (p-1)\,\mathrm{N}_{2,1^p} + \ldots + \mathrm{N}_{p,1^p}\}$$

$$= \frac{n!}{(p+1)!}\{\mathrm{N}_{1,1^p} + 2\mathrm{N}_{2,1^p} + \ldots + p\mathrm{N}_{p,1^p}\};$$

but the sinister is of the form

$$\Sigma \mathrm{N}\,(w_1 w_2) \quad \text{(see Art. 20)}$$

and thus has the value $\frac{1}{2}n!$; hence

$$\mathrm{N}_{1,1^p} + 2\mathrm{N}_{2,1^p} + \ldots + p\mathrm{N}_{p,1^p} = \tfrac{1}{2}(p+1)!,$$

an interesting result.

Art. 29. From a previous result

$$\sum_{a} \mathrm{N}\,(a_1 a_2 \ldots a_m + 1) = (p+1)\,\mathrm{N}_{m,1^p} - \sum_{a} \mathrm{N}\,(a_1 a_2 \ldots a_m 1) = m\mathrm{N}_{m,1^p};$$

hence

$$\sum_{a} \mathrm{N}\,(a_1 a_2 \ldots a_m + 1) = \sum_{a} \mathrm{N}\,(a_1 a_2 \ldots a_{p-m+1} 1) = m\mathrm{N}_{m,1^p};$$

and it may be observed that the numbers, included in

$$\sum_{a} \mathrm{N}\,(a_1 a_2 \ldots a_m + 1),$$

are the conjugates of those included in

$$\sum_{a} \mathrm{N}\,(a_1 a_2 \ldots a_{p-m+1} 1).$$

Art. 30. Also since

$$\sum_{a} \mathrm{N}\,(a_1 a_2 \ldots a_m + 1) = \mathrm{N}_{m,1^{p+1}} - \sum_{a} \mathrm{N}\,(a_1 a_2 \ldots a_{m-1} 1),$$

$$\sum_{a} \mathrm{N}\,(a_1 a_2 \ldots a_{m-1} 1) + \sum_{a} \mathrm{N}\,(a_1 a_2 \ldots a_{p-m+1} 1) = \mathrm{N}_{m,1^{p+1}};$$

and this leads to the relation

$$(p-m+2)\,\mathrm{N}_{p-m+2,1^p} + m\mathrm{N}_{m,1^p} = \mathrm{N}_{m,1^{p+1}}.$$

E.g., for $p = 3$, $m = 2$, $p-m+2 = 3$,

$$3\mathrm{N}_{3,1^3} + 2\mathrm{N}_{2,1^3} = \mathrm{N}_{2,1^4};$$

verified by

$$3.1 + 2.4 = 11.$$

The result is convenient for the calculation of the numbers $N_{m, 1^{p+1}}$ from the numbers $N_{m, 1^p}$.

We have also the remarkable result that the probability of obtaining a permutation, such that the sum of the first m numbers of the descending specification is p, is independent of n, and has the value

$$\frac{(p-m+1)}{(p+1)!} N_{m, 1^p};$$

whenever p is $n-1$ or less.

Art. 31. From the definition we have in respect of the permutations of n numbers

$$N_1 + N_2 + N_3 + \ldots = n!$$

I shall now show that

$$N_{n-\theta+1} + \binom{n-\theta+1}{1} N_{n-\theta+2} + \binom{n-\theta+2}{2} N_{n-\theta+3} + \ldots$$

$$= \sum_{\nu} \frac{n!}{(2!)^{\nu_2}(3!)^{\nu_3}\ldots} \cdot \frac{\theta!}{\nu_1!\,\nu_2!\ldots};$$

the summation being for all values of

$$\nu_1, \nu_2, \ldots$$

such that

$$\Sigma\nu = \theta,$$

$$\Sigma s\nu_s = n.$$

The theorem is the outcome of the multiplication theorem of Art. 12.

Observing that, for all values of s,

$$N(1^s) = 1,$$

we have

$$\frac{(s_1+s_2)!}{s_1!\,s_2!} N(1^{s_1}) N(1^{s_2}) = N(1^{s_1+s_2}) + N(1^{s_1-1}21^{s_2-1}),$$

$$\frac{(s_1+s_2+s_3)!}{s_1!\,s_2!\,s_3!} N(1^{s_1}) N(1^{s_2}) N(1^{s_3}) = N(1^{s_1+s_2+s_3}) + N(1^{s_1+s_2-1}21^{s_3-1})$$
$$+ N(1^{s_1-1}21^{s_2+s_3-1}) + N(1^{s_1-1}21^{s_2-2}21^{s_3-1});$$

and generally, for the product

$$\{N(1)\}^{\nu_1}\{N(1^2)\}^{\nu_2}\ldots\{N(1^p)\}^{\nu_p},$$

since $\Sigma s\nu_s = n$,

$$\frac{n!}{(1!)^{\nu_1}(2!)^{\nu_2}\ldots(p!)^{\nu_p}} = \text{a linear function of numbers } N(\ldots).$$

We may write down a similar result for every permutation of the factors of

$$\{N(1)\}^{\nu_1}\{N(1^2)\}^{\nu_2}\ldots\{N(1^p)\}^{\nu_p}$$

and, by addition, obtain

$$\frac{n!}{(1!)^{\nu_1}(2!)^{\nu_2}\ldots(p!)^{\nu_p}} \cdot \frac{\theta!}{\nu_1!\,\nu_2!\ldots\nu_p!} = \text{linear function of numbers N},$$

where $\Sigma\nu = \theta$,

Further, we obtain a result of this nature for all values of

$$\nu_1, \nu_2, \ldots \nu_p,$$

such that $\Sigma s\nu_s = n$, $\Sigma\nu = \theta$; and, by addition, we obtain

$$\Sigma \frac{n!}{(1!)^{\nu_1}(2!)^{\nu_2}\ldots(p!)^{\nu_p}} \cdot \frac{\theta!}{\nu_1!\,\nu_2!\ldots\nu_p!} = \text{linear function of numbers N},$$

where $\Sigma s\nu_s = n$, $\Sigma\nu = \theta$.

We have now to determine the linear function of numbers N which appears on the dexter.

If one such number be

$$\text{N}(abc\ldots),$$

it is evident that

$$(abc\ldots)$$

is some composition of the number n.

Consider the product of θ factors

$$\text{N}(1^{s_1})\,\text{N}(1^{s_2})\ldots\text{N}(1^{s_\theta}),$$

where $\Sigma s = n$.

The process of multiplication produces N numbers of θ different kinds.

In the first place we throw all the units together,

$$\text{N}(1^{s_1+s_2+\ldots+s_\theta}),$$

one N number containing n parts.

In the second place we combine a consecutive pair of factors and throw the remainder of the units together, thus producing $\theta-1$ N numbers each containing $n-1$ parts, viz.,

$$\text{N}(1^{s_1-1}21^{s_2-1+s_3+\ldots+s_\theta}),$$
$$\text{N}(1^{s_1+s_2-1}21^{s_3-1+s_4+\ldots+s_\theta}),$$
$$\vdots$$
$$\text{N}(1^{s_1+s_2+\ldots+s_{\theta-1}-1}21^{s_\theta-1}).$$

In the third place we combine two consecutive pairs (including, of course, a consecutive three) of factors and throw the remainder of the units together, thus producing

$$\binom{\theta-1}{2}$$

N numbers each containing $n-2$ parts, viz., the series of which one is

$$\text{N}(1^{s_1-1}21^{s_2-2}21^{s_3-1+s_4+\ldots+s_\theta}).$$

Notice that, if $s_2 = 1$, this becomes

$$\text{N}(1^{s_1-1}31^{s_3-1+s_4+\ldots+s_\theta}).$$

We proceed in this manner until finally we combine $\theta-1$ consecutive pairs and throw the remainder of the units together, thus producing

$$\binom{\theta-1}{\theta-1}$$

N numbers, each containing $n-\theta+1$ parts.

Hence the compositions that present themselves are included in those enumerated by

$$N_n,\ N_{n-1},\ \ldots,\ N_{n-\theta+1}.$$

We have to consider the product

$$N(1^{s_1})N(1^{s_2})\ldots N(1^{s_\theta})$$

in all of its permutations and for every system of values of

$$s_1,\ s_2,\ \ldots,\ s_\theta,$$

such that

$$s_1+s_2+\ldots+s_\theta = n.$$

Hence, from considerations of symmetry, and attending to the *modus operandi* of the multiplication theorem, we find that the whole of the compositions enumerated by

$$N_n,\ N_{n-1},\ \ldots,\ N_{n-\theta+1}$$

present themselves.

Hence the linear function we seek is a linear function of

$$N_{n-\theta+1},\ N_{n-\theta+2},\ \ldots,\ N_{n-1},\ N_n,$$

and it remains to determine the coefficients.

The number of products, including permutations,

$$N(1^{s_1})N(1^{s_2})\ldots N(1^{s_\theta}),$$

which we have to consider, is equal to the numbers of compositions of n into θ parts, viz., it is

$$\binom{n-1}{n-\theta};$$

each of these produces

$$\binom{\theta-1}{m}$$

N numbers, each containing $n-m$ parts.

There are thus

$$\binom{n-1}{n-\theta}\binom{\theta-1}{m}$$

N numbers, each containing $n-m$ parts.

But there are only

$$\binom{n-1}{m}$$

different N numbers, each containing $n-m$ parts, because

$$\binom{n-1}{m}$$

is equal to the number of compositions of n into $n-m$ parts.

Hence, each N number, comprised in

$$N_{n-m},$$

will occur

$$\frac{\binom{n-1}{n-\theta}\binom{\theta-1}{m}}{\binom{n-1}{m}} = \binom{n-m-1}{n-\theta} \text{ times.}$$

Hence the required linear function is

$$\Sigma \binom{n-m-1}{n-\theta} N_{n-m},$$

or

$$N_{n-\theta+1} + \binom{n-\theta+1}{1} N_{n-\theta+2} + \binom{n-\theta+2}{2} N_{n-\theta+3} + \ldots + \binom{n-1}{\theta-1} N_n,$$

and the final result is

$$\Sigma \frac{n!}{(1!)^{\nu_1}(2!)^{\nu_2}\ldots(p!)^{\nu_p}} \cdot \frac{\theta!}{\nu_1!\nu_2!\ldots\nu_p!}$$

$$= N_{n-\theta+1} + \binom{n-\theta+1}{1} N_{n-\theta+2} + \binom{n-\theta+2}{2} N_{n-\theta+3} + \ldots + \binom{n-1}{\theta-1} N_n,$$

where

$$\Sigma s\nu_s = n, \ \Sigma\nu = \theta.$$

Part II.—Section 3.

Art. 32. In the preceding pages we have had under view the permutations of n different numbers. As I am now taking in hand the general case of numbers which possess any number of similarities, I find it convenient to slightly alter the point of view.

Let

$$\alpha, \beta, \gamma, \ldots$$

denote numbers in descending order of magnitude, and suppose there are

p number equal to α,

q ,, ,, β,

r ,, ,, γ.

so that, placed in descending order, the assemblages may be written

$$\alpha^p\beta^q\gamma^r\ldots.$$

I say that the assemblage is specified by the composition

$$(pqr\ldots).$$

As equalities may occur between the numbers p, q, r, ..., I take, for greater generality, the specifying composition

$$(p_1^{\pi_1}p_2^{\pi_2}\ldots).$$

It will be seen later that the order of occurrence of the parts of this composition is immaterial, so that we may consider the parts p_1, p_2, ... to be in descending order of magnitude and the specification to be denoted by a partition

$$(p_1^{\pi_1}p_2^{\pi_2}\ldots).$$

E.g., we obtain the same results for each of the six assemblages,

$$\alpha\alpha\alpha\beta\beta\gamma,\quad \alpha\alpha\alpha\beta\gamma\gamma,\quad \alpha\alpha\beta\beta\beta\gamma,$$
$$\alpha\alpha\beta\gamma\gamma\gamma,\quad \alpha\beta\beta\beta\gamma\gamma,\quad \alpha\beta\beta\gamma\gamma\gamma,$$

the specification of each assemblage being

$$(321).$$

Every permutation has a descending specification.

E.g.,

$$\alpha\beta\alpha\alpha\gamma\beta$$

has the descending specification

$$(231).$$

In the case considered in Part I. the assemblage of numbers had the specification

$$(1^n)$$

since there were no similarities, and the numbers N (...) were expressed in terms of the coefficients obtained by the multinomial expansion

$$(\alpha_1+\alpha_2+\alpha_3+\ldots)^n.$$

E.g., we found

$\mathrm{N}(a)$ = coefficient of symmetric function (a) in the expansion,

$\mathrm{N}(ab)+\mathrm{N}(a+b)$ = ,, ,, (ab) ,, ,,

where, in the first case, $a = n$, and in the second, $a+b = n$.

In a usual notation let

$$h_1,\ h_2,\ h_3,\ \ldots$$

denote the homogeneous product sums, of the successive orders, of the roots of the equation

$$x^n - a_1x^{n-1} + a_2x^{n-2} - a_3x^{n-3} + \ldots = 0\,;$$

we may say that, in Part I., the auxiliary generating function was

$$(\alpha_1+\alpha_2+\alpha_3+\ldots)^n = h_1^n,$$

α_1, α_2, α_3, ... being the roots of the equation.

Art. 33. In the present case the auxiliary generating function is

$$h_{p_1}^{\pi_1}h_{p_2}^{\pi_2}h_{p_3}^{\pi_3}\ldots,$$

as will appear.

For it was shown, *loc. cit.*, that the number of ways of distributing the objects, as specified, into different parcels containing a, b, c... objects respectively is the coefficient of the symmetric function

$$(abc\ldots)$$

in the development of the symmetric function

$$h_{p_1}^{\pi_1} h_{p_2}^{\pi_2} h_{p_3}^{\pi_3} \ldots$$

as a sum of monomial symmetric functions.

Let this coefficient be denoted by

$$\mathrm{C}\,(abc\ldots),$$

and let the number of arrangements of the objects, which have a descending specification

$$(abc\ldots),$$

be denoted by

$$\mathrm{N}\,(abc\ldots).$$

Let the whole number of objects be

$$\Sigma \pi p = n.$$

Then, when $a = n$, clearly

$$\mathrm{N}\,(a) = \mathrm{C}\,(a) = 1,$$

and when $a+b = n$, $\mathrm{C}\,(ab)$ is the number of arrangements into two different parcels containing a, b objects respectively, and by previous reasoning

$$\mathrm{N}\,(ab) + \mathrm{N}\,(a+b) = \mathrm{C}\,(ab)\,;$$

and, when $a+b+c = n$,

$$\mathrm{N}\,(abc) + \mathrm{N}\,(a+b,\, c) + \mathrm{N}\,(a,\, b+c) + \mathrm{N}\,(a+b+c) = \mathrm{C}\,(abc),$$

and so forth as in the simple case already considered.

Hence

$$\begin{aligned} \mathrm{N}\,(ab) &= \mathrm{C}\,(ab) - \mathrm{C}\,(a+b), \\ \mathrm{N}\,(abc) &= \mathrm{C}\,(abc) - \mathrm{C}\,(a+b,\, c) - \mathrm{C}\,(a,\, b+c) + \mathrm{C}\,(a+b+c), \\ \mathrm{N}\,(abcd) &= \mathrm{C}\,(abcd) - \mathrm{C}\,(a+b,\, c,\, d) - \mathrm{C}\,(a,\, b+c,\, d) - \mathrm{C}\,(a,\, b,\, c+d) \\ &\quad + \mathrm{C}\,(a+b,\, c+d) + \mathrm{C}\,(a,\, b+c+d) + \mathrm{C}\,(a+b+c,\, d) \\ &\quad - \mathrm{C}\,(a+b+c+d), \end{aligned}$$

&c.,

the numbers N being all expressible in terms of coefficients of the auxiliary generating function.

Art. 34. *E.g.* Take objects $\alpha\alpha\alpha\beta\beta\gamma$, where α, β, γ are in descending order of magnitude.

Since

$$\begin{aligned} h_3 h_2 h_1 = {} & (6) + 3\,(51) + 5\,(42) + 8\,(41^2) + 6\,(3^2) + 12\,(321) \\ & + 19\,(31^3) + 15\,(2^3) + 24\,(2^2 1^2) + 38\,(21^4) + 60\,(1^6), \end{aligned}$$

we calculate, from the above formulæ,

$$N(6) = 1,$$

$$N(51) = 3-1 = 2,$$

$$N(3^2) = 6-1 = 5,$$

$$N(321) = 12-5-2-1 = 4$$

and so on.

The five arrangements, enumerated by $N(3^2)$, are

$$\alpha\alpha\beta\alpha\beta\gamma$$
$$\alpha\alpha\gamma\alpha\beta\beta$$
$$\alpha\beta\gamma\alpha\alpha\beta$$
$$\alpha\beta\beta\alpha\alpha\gamma$$
$$\beta\beta\gamma\alpha\alpha\alpha,$$

each having the descending specification (3^2).

The four arrangements, enumerated by $N(321)$, are

$$\alpha\alpha\beta\alpha\gamma\beta$$
$$\alpha\alpha\gamma\beta\beta\alpha$$
$$\alpha\beta\gamma\alpha\beta\alpha$$
$$\alpha\beta\beta\alpha\gamma\alpha,$$

each having the descending specification (321).

The complete results *quâ* numbers specified by (321) are

$N(31^3)$, $N(1^33)$, $N(21^4)$, $N(1^42)$, $N(2112)$, $N(121^3)$, $N(1^321)$, $N(1^221^2)$, $N(1^6)$	each 0	. .	0
$N(6)$, $N(41^2)$, $N(1^24)$, $N(131^2)$, $N(1^231)$, $N(2^21^2)$, $N(1^22^2)$, $N(2121)$, $N(1212)$	,, 1	. .	9
$N(51)$, $N(15)$, $N(312)$, $N(213)$	,, 2	. .	8
$N(141)$, $N(1221)$	,, 3	. .	6
$N(42)$, $N(24)$, $N(321)$, $N(123)$	,, 4	. .	16
$N(3^2)$, $N(231)$, $N(132)$	,, 5	. .	15
$N(2^3)$	,, 6	. .	6
			60

60 being, of course, the total number of permutations of the objects.

Art. 35. The method of calculation establishes that the number N (...) is unaltered by reversal of the order of the numbers in the bracket.

Also that the results are only dependent upon the magnitudes of the parts in the specification of the assemblage and not upon the order of their occurrence.

General Investigation of a Generating Function.

Art. 36. I have shown above that, for numbers specified by

$$(p_1^{\pi_1} p_2^{\pi_2}\ldots),$$

an auxiliary generating function is

$$h_{p_1}^{\pi_1} h_{p_2}^{\pi_2}\ldots,$$

for, from its expansion in terms of monomial symmetric functions, the numbers

$$\text{N}\,(abc\ldots)$$

can be successively calculated.

For present convenience I take the above generating function to be

$$h_p h_q h_r\ldots$$

and recall that

$$\text{N}\,(abc\ldots)+\text{N}\,(a+b,\ c,\ \ldots)+\text{N}\,(a,\ b+c,\ \ldots)+\ldots$$

is equal to the coefficient of symmetric function

$$(abc\ldots)$$

in the expansion of

$$h_p h_q h_r\ldots.$$

The above linear function of the numbers

$$\text{N}\,(\ldots)$$

is formed by adding *adjacent* numbers

$$0,\ 1,\ 2,\ 3,\ \ldots,\ k \text{ at a time,}$$

where the numbers a, b, c, ... are k in number.

It thus comprises 2^{k-1} terms in general.

Art. 37. Let this linear function be denoted by

$$\theta_{\text{N}}\{(a)(b)(c)\ldots\},$$

so that if we write

$$h_p h_q h_r\ldots = \Sigma\text{C}\,(abc\ldots).(abc\ldots),$$

$$\theta_{\text{N}}\{(a)(b)(c)\ldots\} = \text{C}\,(abc\ldots).$$

From this system of linear relations is determined the set

$\text{N}\,(a) = \text{C}\,(a)$, where $a = n$,

$\text{N}\,(ab) = \text{C}\,(ab) - \text{C}\,(a+b)$, where $a+b = n$,

$\text{N}\,(abc) = \text{C}\,(abc) - \text{C}\,(a+b,\ c) - \text{C}\,(a,\ b+c) + \text{C}\,(a+b+c)$, where $a+b+c = n$, and so on;

the law of formation of the linear functions of the numbers

$$C(\ldots)$$

being similar to that which occurs in

$$\theta_N\{(a)(b)(c)\ldots\},$$

with the exception that the signs are alternately positive and negative, depending upon the numbers of integers in the brackets.

Art. 38. Denote this linear function of the numbers $C(\ldots)$ by

$$\phi_C\{(a)(b)(c)\ldots\},$$

so that

$$N(abc\ldots) = \phi_C\{(a)(b)(c)\ldots\}.$$

When it is necessary to put in evidence the numbers whose permutations are under examination we may write the two formulæ

$$\theta_N\{(a)(b)(c)\ldots\}_{(pqr\ldots)} = C(abc\ldots)_{(pqr\ldots)};$$

$$N(abc\ldots)_{(pqr\ldots)} = \phi_C\{(a)(b)(c)\ldots\}_{(ppr\ldots)}.$$

SECTION 4.

Digression on the Forms θ_N, ϕ_C.

Art. 39. Define in general, so that

$$\theta_N\{(a_1\ldots a_{s-1}a_s)(b_1b_2\ldots b_{t-1}b_t)(c_1c_2\ldots c_{u-1}c_u)(d_1d_2\ldots d_v)\ldots(k_1k_2\ldots k_z)\},$$

where there are k symbols $a, b, c, d, \ldots, k$, denotes the 2^{k-1} terms forming the series

$$\begin{aligned}
&N(a_1\ldots k_z)\\
&+N(a_1\ldots a_{s-1},\ a_s+b_1,\ b_2\ldots k_z)\\
&+N(a_1\ldots b_{t-1},\ b_t+c_1,\ c_2\ldots k_z)\\
&+\ldots\\
&+N(a_1\ldots a_{s-1},\ a_s+b_1,\ b_2\ldots b_{t-1},\ b_t+c_1,\ c_2\ldots k_z)\\
&+\ldots,
\end{aligned}$$

where additions take place,

$0, 1, 2, \ldots, k-1$ at a time between the pairs a_s, b_1; b_t, c_1; c_u, d_1; ....

Art. 40. Similarly define

$$\phi_C\{(a_1\ldots a_{s-1}a_s)(b_1b_2\ldots b_{t-1}b_t)(c_1c_2\ldots c_{u-1}c_u)(d_1d_2\ldots d_v)\ldots(k_1k_2\ldots k_z)\}$$

to denote the 2^{k-1} terms forming the series

$$\begin{aligned}
&C(a_1\ldots k_z)\\
&-C(a_1\ldots a_{s-1},\ a_s+b_1,\ b_2\ldots k_z)\\
&-C(a_1\ldots b_{t-1},\ b_t+c_1,\ c_2\ldots k_z)\\
&-\ldots\\
&+C(a_1\ldots a_{s-1},\ a_s+b_1,\ b_2\ldots b_{t-1},\ b_t+c_1,\ c_2\ldots k_z)\\
&+\ldots,
\end{aligned}$$

formed according to the same law, but the successive blocks of terms having alternately positive and negative signs.

Art. 41. I proceed to generalise the two results

$$\theta_N\{(a)(b)(c)\ldots\} = \phi_C(abc\ldots),$$
$$\theta_N(abc\ldots) = \phi_C\{(a)(b)(c)\ldots\}.$$

By definition

$$\theta_N\{(a_1\ldots a_{s-1}a_s)(b_1b_2\ldots b_t)\}$$
$$= N(a_1\ldots b_t) + N(a_1\ldots a_{s-1},\ a_s+b_1,\ b_2\ldots b_t);$$

and since

$$N(abc\ldots) = \theta_N(abc\ldots) = \phi_C\{(a)(b)(c)\ldots\},$$

this

$$= \phi_C\{(a_1)(a_2)\ldots(b_t)\} + \phi_C\{(a_1)\ldots(a_{s-1})(a_s+b_1)(b_2)\ldots(b_t)\}.$$

Now the sum of these two terms is precisely

$$\phi_C\{(a_1)(a_2)\ldots(a_{s-1})(a_sb_1)(b_2)\ldots(b_t)\},$$

because the terms involving

$$a_s+b_1$$

in

$$\phi_C\{(a_1)(a_2)\ldots(b_t)\}$$

are the same, with opposite sign, as those involved in

$$\phi_C\{(a_1)\ldots(a_{s-1})(a_s+b_1)(b_2)\ldots(b_t)\},$$

and therefore cancel them.

Hence the result

$$\theta_N\{(a_1\ldots a_{s-1}a_s)(b_1b_2\ldots b_t)\} = \phi_C\{(a_1)(a_2)\ldots(a_{s-1})(a_sb_1)(b_2)\ldots(b_t)\}.$$

Art. 42. Again

$$\theta_N\{(a_1\ldots a_s)(b_1\ldots b_t)(c_1\ldots c_u)\}$$
$$= N(a_1\ldots c_u) + N(a_1\ldots a_{s-1},\ a_s+b_1,\ b_2\ldots c_u)$$
$$+ N(a_1\ldots b_{t-1},\ b_t+c_1,\ c_2\ldots c_u)$$
$$+ N(a_1\ldots a_{s-1},\ a_s+b_1,\ b_2\ldots b_{t-1},\ b_t+c_1,\ c_2\ldots c_u),$$
$$= \theta_N(a_1\ldots c_u) + \theta_N(a_1\ldots a_{s-1},\ a_s+b_1,\ b_2\ldots c_u)$$
$$+ \theta_N(a_1\ldots b_{t-1},\ b_t+c_1,\ c_2\ldots c_u) + \theta_N(a_1\ldots a_{s-1},\ a_s+b_1,\ b_2\ldots b_{t-1},\ b_t+c_1,\ c_2\ldots c_u),$$
$$= \phi_C\{(a_1)\ldots(c_u)\} + \phi_C\{(a_1)\ldots(a_{s-1})(a_s+b_1)(b_2)\ldots(c_u)\}$$
$$+ \phi_C\{(a_1)\ldots(b_{t-1})(b_t+c_1)(c_2)\ldots(c_u)\} + \phi_C\{(a_1)\ldots(a_s+b_1)\ldots(b_t+c_1)\ldots(c_u)\},$$
$$= \phi_C\{(a_1)\ldots(a_{s-1})(a_sb_1)(b_2)\ldots(c_u)\}$$
$$+ \phi_C\{(a_1)\ldots(a_{s-1})(a_sb_1)(b_2)\ldots(b_t+c_1)(c_2)\ldots(c_u)\},$$
$$= \phi_C\{(a_1)\ldots(a_{s-1})(a_sb_1)(b_2)\ldots(b_{t-1})(b_tc_1)(c_2)\ldots(c_u)\},$$

by successive use of the formula Art. 41 above.

Also, clearly, if $t = 1$

$$\theta_{\rm N}\{(a_1 \ldots a_s)(b_1)(c_1 \ldots c_u)\}$$
$$= \phi_{\rm C}\{(a_1)\ldots(a_{s-1})(a_s b_1 c_1)(c_2)\ldots(c_u)\}.$$

Art. 43. Therefore, by induction, we can express any form

$$\theta_{\rm N}\{\quad\}$$

as a form

$$\phi_{\rm C}\{\quad\}.$$

The law is well seen by a particular case, viz.,

$$\theta_{\rm N}\{(a)(b)(c)(d)\} = \phi_{\rm C}(abcd),$$
$$\theta_{\rm N}\{(ab)(c)(d)\} = \phi_{\rm C}\{(a)(bcd)\},$$
$$\theta_{\rm N}\{(a)(bc)(d)\} = \phi_{\rm C}\{(ab)(cd)\},$$
$$\theta_{\rm N}\{(a)(b)(cd)\} = \phi_{\rm C}\{(abc)(d)\},$$
$$\theta_{\rm N}\{(a)(bcd)\} = \phi_{\rm C}\{(ab)(c)(d)\},$$
$$\theta_{\rm N}\{(ab)(cd)\} = \phi_{\rm C}\{(a)(bc)(d)\},$$
$$\theta_{\rm N}\{(abc)(d)\} = \phi_{\rm C}\{(a)(b)(cd)\},$$
$$\theta_{\rm N}\{(abcd)\} = \phi_{\rm C}\{(a)(b)(c)(d)\}.$$

We have, in respect of the four letters, $8 = 2^3$ relations; the letters always occur in the order

$$a, b, c, d,$$

and to obtain the form $\phi_{\rm C}\{\quad\}$, which is equated to a form $\theta_{\rm N}\{\quad\}$, we may make use of the zig-zag conjugate law; *e.g.*, connect with

$$(ab)\,(cd)$$

the composition 22; take the zig-zag conjugate of this, viz., 121, and then write

$$\theta_{\rm N}\{(ab)(cd)\} = \phi_{\rm C}\{(a)(bc)(d)\},$$

and

$$\theta_{\rm N}\{(a)(bc)(d)\} = \phi_{\rm C}\{(ab)(cd)\};$$

and so in every case.

Art. 44. In the general case of p letters we obtain 2^{p-1} relations corresponding to the 2^{p-1} compositions of p; the relations are obtainable from zig-zag conjugation of such compositions and, in any relation

$$\theta_{\rm N}\{\quad\} = \phi_{\rm C}\{\quad\},$$

we may interchange the form-symbols

$$\theta_{\rm N},\ \phi_{\rm C}.$$

Art. 45. In the above investigation we obtained incidentally certain linear relations between the forms

$$\theta_{\rm N},$$

and also between the forms ϕ_C,

which must now be set forth in a regular manner.

The former relations are of the type

$$\begin{aligned}
&\theta_N\{(a_1\ldots a_s)(b_1\ldots b_t)(c_1\ldots c_u)\ldots\}\\
= \;&\theta_N(a_1\ldots a_s b_1\ldots b_t c_1\ldots c_u\ldots)\\
&+\theta_N(a_1\ldots a_{s-1},\ a_s+b_1,\ b_2\ldots b_t c_1\ldots c_u\ldots)\\
&+\theta_N(a_1\ldots a_s b_1\ldots b_{t-1},\ b_t+c_1,\ c_2\ldots c_u\ldots)\\
&+\ldots\\
&+\theta_N(a_1\ldots a_{s-1},\ a_s+b_1,\ b_2\ldots b_{t-1},\ b_t+c_1,\ c_2\ldots c_u\ldots)\\
&+\ldots;
\end{aligned}$$

this follows directly from the definition of the form $\theta_N\{\ \}$, since

$$\theta_N(abc\ldots) = N(abc\ldots).$$

Art. 46. The latter relations are of the type

$$\begin{aligned}
&\phi_C\{(a_1\ldots a_3)(b_1\ldots b_t)(c_1\ldots c_u)\ldots\}\\
= \;&\phi_C(a_1\ldots a_s b_1\ldots b_t c_1\ldots c_u\ldots)\\
&-\phi_C(a_1\ldots a_{s-1},\ a_s+b_1,\ b_2\ldots b_t c_1\ldots c_u\ldots)\\
&-\phi_C(a_1\ldots a_s b_1\ldots b_{t-1},\ b_t+c_1,\ c_2\ldots c_u\ldots)\\
&-\ldots\\
&+\phi_C(a_1\ldots a_{s-1},\ a_s+b_1,\ b_2\ldots b_{t-1},\ b_t+c_1,\ c_2\ldots c_u\ldots)\\
&+\ldots;
\end{aligned}$$

which also follows directly from the definition of the form $\phi_C\{\ \}$, since

$$\phi_C(abc\ldots) = C(abc\ldots).$$

Art. 47. We have other linear relations of the type

$$\begin{aligned}
&\theta_N\{(a_1\ldots a_s)(b_1\ldots b_t)(c_1\ldots c_u)\}\\
= \;&\theta_N\{(a_1\ldots b_t)(c_1\ldots c_u)\}\\
&+\theta_N\{(a_1\ldots a_{s-1},\ a_s+b_1,\ b_2\ldots b_t)(c_1\ldots c_u)\};
\end{aligned}$$

$$\begin{aligned}
&\phi_C\{(a_1\ldots a_s)(b_1\ldots b_t)(c_1\ldots c_u)\}\\
= \;&\phi_C\{(a_1\ldots b_t)(c_1\ldots c_u)\}\\
&-\phi_C\{(a_1\ldots a_{s-1},\ a_s+b_1,\ b_2\ldots b_t)(c_1\ldots c_u)\}.
\end{aligned}$$

In fact, the law may be taken to operate as between any sets of consecutive factors in $\phi_N\{\ \}$ and $\phi_C\{\ \}$ respectively, leaving the remaining factors untouched.

Thus it is easy to verify the three relations

$$\begin{aligned}
&\theta_N\{(ab)(cd)(ef)(gh)\} \\
= \; &\theta_N\{(abcd)(ef)(gh)\} \\
&+\theta_N\{(a,\ b+c,\ d)(ef)(gh)\}, \\
= \; &\theta_N\{(ab)(cd)(efgh)\} \\
&+\theta_N\{(ab)(cd)(e,\ f+g,\ h)\}, \\
= \; &\theta_N\{(abcd)(efgh)\} \\
&+\theta_N\{(abcd)(e,\ f+g,\ h)\} \\
&+\theta_N\{(a,\ b+c,\ d)(efgh)\} \\
&+\theta_N\{(a,\ b+c,\ d)(e,\ f+g,\ h)\}\,;
\end{aligned}$$

and the further three

$$\begin{aligned}
&\phi_C\{(ab)(cd)(ef)(gh)\} \\
= \; &\phi_C\{(abcd)(ef)(gh)\} \\
&-\phi_C\{(a,\ b+c,\ d)(ef)(gh)\}, \\
= \; &\phi_C\{(ab)(cd)(efgh)\} \\
&-\phi_C\{(ab)(cd)(e,\ f+g,\ h)\}, \\
= \; &\phi_C\{(abcd)(efgh)\} \\
&-\phi_C\{(a,\ b+c,\ d)(efgh)\} \\
&-\phi_C\{(abcd)(e,\ f+g,\ h)\} \\
&+\phi_C\{(a,\ b+c,\ d)(e,\ f+g,\ h)\}.
\end{aligned}$$

Art. 48. From these relations we may obtain new relations by transforming from θ_N to ϕ_C, or *vice versâ*.

Thus from relations of type

$$\theta_N\{(a)(b)(c)\} = \theta_N(abc)+\theta_N(a+b,\ c)+\theta_N(a,\ b+c)+\theta_N(a+b+c),$$

we obtain those of type

$$\phi_C(abc) = \phi_C\{(a)(b)(c)\}+\phi_C\{(a+b)(c)\}+\phi_C\{(a)(b+c)\}+\phi_C(a+b+c)\,;$$

and from those of type

$$\phi_C\{(a)(b)(c)\} = \phi_C(abc)-\phi_C(a+b,\ c)-\phi_C(a,\ b+c)+\phi_C(a+b+c),$$

we obtain others of type

$$\theta_N(abc) = \theta_N\{(a)(b)(c)\}-\theta_N\{(a+b)(c)\}-\theta_N\{(a)(b+c)\}+\theta_N(a+b+c).$$

These new expressions for

$$\theta_N(abc\ldots) \quad \text{and} \quad \phi_C(abc\ldots),$$

with an obviously analogous law to that we have frequently met with, are of great importance.

From the relation

$$\phi_C\{(a)(b)(c)(d)\} = \phi_C\{(ab)(cd)\}-\phi_C\{(a+b)(cd)\}-\phi_C\{(ab)(c+d)\}+\phi_C\{(a+b)(c+d)\},$$

we obtain

$$\theta_N(abcd) = \theta_N\{(a)(bc)(d)\}-\theta_N\{(a+b,\ c)(d)\}-\theta_N\{(a)(b,\ c+d)\}+\theta_N\{(a+b,\ c+d)\};$$

and there is no necessity to give further examples.

SECTION 5.

Art. 49. The differential operator, of order s, that is so frequently of use in the theory of symmetric functions, viz. :—

$$\frac{1}{s!}(\partial_{a_1}+a_1\partial_{a_2}+a_2\partial_{a_3}+\ldots)^s = D_s,$$

can now be employed.

Remembering that operating upon monomial symmetric functions,

$$D_a(a) = 1,$$
$$D_a(b) = 0 \text{ unless } b = a,$$
$$D_aD_bD_c\ldots(abc\ldots) = 1;$$

and generally that D_a obliterates a number a from the partition of a function and causes it to vanish if no such number presents itself, it is clear that

$$D_aD_bD_c\ldots h_ph_qh_r\ldots = C(abc\ldots)_{(pqr\ldots)};$$

and thence if we write

$$\phi_D\{(a)(b)(c)\ldots\} = D_aD_bC_c\ldots-D_{a+b}D_c\ldots-D_aD_{b+c}\ldots-\ldots$$

according to a law *derivable* from that which defines

$$\phi_C\{(a)(b)(c)\ldots\} \text{ (see Art. 38)},$$

we find

$$N(abc\ldots)_{(pqr\ldots)} = \phi_D\{(a)(b)(c)\ldots\}h_ph_qh_r\ldots.$$

Art. 50. Observe that in the paper to which reference has been made it was shown that

$$C(abc\ldots)_{pqr\ldots} = C(pqr\ldots)_{abc\ldots}.$$

Two consequences flow from this fact.

Firstly

$$\theta_N\{(a)(b)(c)\ldots\}_{(pqr\ldots)} = \theta_N\{(p)(q)(r)\ldots\}_{(abc\ldots)},$$

which is a theorem of reciprocity for the numbers

$$N(\ldots).$$

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Secondly, since

$$D_a D_b D_c \ldots h_p h_q h_r \ldots = D_p D_q D_r \ldots h_a h_b h_c \ldots,$$

$$N(abc\ldots)_{(pqr\ldots)} = D_p D_q D_r \ldots (h_a h_b h_c \ldots - h_{a+b} h_c \ldots - h_a h_{b+c} \ldots - \ldots);$$

where, on the dexter, the operand is a function formed from the functions $h_1, h_2, h_3, \ldots$ in the same manner as

$$\phi_D\{(a)(b)(c)\ldots\}$$

is formed from the operators

$$D_1, D_2, D_3, \ldots.$$

Hence

$$N(abc\ldots)_{(pqr\ldots)} = D_p D_q D_r \ldots \phi_h\{(a)(b)(c)\ldots\},$$

where

$$\phi_h\{(a)(b)(c)\ldots\} = h_a h_b h_c \ldots - h_{a+b} h_c \ldots - h_a h_{b+c} \ldots - \ldots.$$

Art. 51. I now write

$$\phi_h\{(a)(b)(c)\ldots\} = h_{abc\ldots};$$

so that

$$N(abc\ldots)_{(pqr\ldots)} = D_p D_q D_r \ldots h_{abc\ldots};$$

and it appears that

$$h_{abc\ldots}$$

is the true generating function of the numbers

$$N(abc\ldots)$$

for the permutations of assemblages of numbers of *all specifications.*

In fact,

$$h_{abc\ldots} = \Sigma N(abc\ldots)_{(pqr\ldots)} \cdot (pqr\ldots);$$

and the expansion of

$$h_{abc\ldots}$$

as a linear function of monomial symmetric functions gives a complete account of numbers

$$N(abc\ldots).$$

Art. 52. Before proceeding to a rapid examination of this new and most important symmetric function

$$h_{abc\ldots},$$

never before I believe introduced into algebraic analysis, I give complete tables of the numbers N (...) as far as $n = 2$.

$n = 2.$

	(2)	(1^2)
N (2)	1	1
N (1^2)		1

= specification.

$n = 3.$

	(3)	(21)	(1^3)
N (3)	1	1	1
N (21)		1	2
N (1^3)			1

= specification.

$n = 4.$

	(4)	(31)	(2^2)	(21^2)	(1^4)
N (4)	1	1	1	1	1
N (31)		1	1	2	3
N (2^2)		1	2	3	5
N (121)			1	2	5
N (21^2)				1	3
N (1^4)					1

= specification.

$n = 5.$

	(5)	(41)	(32)	(31^2)	(2^21)	(21^3)	(1^5)
N (5)	1	1	1	1	1	1	1
N (41)		1	1	2	2	3	4
N (32)		1	2	3	4	6	9
N (131)			1	2	3	6	11
N (2^21)			1	2	4	8	16
N (31^2)				1	1	3	6
N (212)				1	2	5	11
N (121^2)					1	3	9
N (21^3)						1	4
N (1^5)							1

= specification.

$n = 6.$

(Header row) = {specification.

	(6)	(51)	(42)	(3^2)	(41^2)	(321)	(2^3)	(31^3)	(2^21^2)	(21^4)	(1^6)
N (6)	1	1	1	1	1	1	1	1	1	1	1
N (51)		1	1	1	2	2	2	3	3	4	5
N (42)		1	2	2	3	4	5	6	7	10	14
N (3^2)		1	2	3	3	5	6	7	9	13	19
N (141)			1	1	2	3	4	6	7	12	19
N (231)			1	2	2	5	7	9	13	23	40
N (312)					1	2	3	5	7	14	26
N (321)			1	1	2	4	6	8	11	20	35
N (2^3)			1	2	2	6	10	11	18	33	61
N (41^2)					1	1	1	3	3	6	10
N (31^3)								1	1	4	10
N (12^21)				1		3	6	6	13	28	61
N (2^21^2)						1	2	3	6	15	35
N (131^2)						1	2	3	5	12	26
N (2121)						1	3	3	7	17	40
N (21^22)								1	2	7	19
N (1^221^2)							1		2	6	19
N (121^3)									1	4	14
N (21^4)										1	5
N (1^6)											1

To explain—it will be found that

$$h_{131} = h_3h_1{}^2 - 2h_4h_1 + h_5,$$
$$= (32) + 2(31^2) + 3(2^21) + 6(21^3) + 11(1^5),$$

corresponding to row 4 of the table for $n = 5$.

Art. 53. Another symmetric function

$$a_{p_1p_2p_3\ldots}$$

is formed from the elements

$$a_1, a_2, a_3, \ldots$$

in the same manner as the symmetric function

$$h_{p_1p_2p_3\ldots}$$

from the elements

$$h_1, h_2, h_3, \ldots .$$

Section 6.

The Symmetric Functions $h_{p_1 p_2 p_3 \ldots}$, $a_{p_1 p_2 p_3 \ldots}$.

Art. 54. These two new functions are of fundamental importance, not only in this investigation, but in the theory of symmetric functions generally.

In regard to the algebraic equation

$$x^n - a_1 x^{n-1} + a_2 x^{n-2} - \ldots = 0,$$

$h_1, h_2, h_3, \ldots$ are the homogeneous product sums of the roots and the two sets of elements

$$a_1, a_2, a_3, \ldots,$$
$$h_1, h_2, h_3, \ldots$$

have reciprocal properties which it is useful to briefly glance at.

We have

$$h_1 = a_1 = (1),$$
$$h_2 = a_1^2 - a_2 = (2) + (1^2),$$
$$h_3 = a_1^3 - 2a_1 a_2 + a_3 = (3) + (21) + (1^3),$$

and, in general,

$$h_n = \Sigma (-)^{n+\Sigma k} \frac{(\Sigma k)!}{k_1! k_2! \ldots k_s!} a_1^{k_1} a_2^{k_2} \ldots a_s^{k_s}.$$

The two series of elements are connected in such wise that, in any relation between the elements, the symbols a, h may be interchanged. Thus, from

$$a_1^2 - 3a_2 = -2h_1^2 + 3h_2$$

is derived

$$h_1^2 - 3h_2 = -2a_1^2 + 3a_2.$$

As a particular case it is found that a_s is the same function of the elements $h_1, h_2, h_3, \ldots$ that h_s is of the elements $a_1, a_2, a_3, \ldots$.

If functions of the elements $h_1, h_2, h_3, \ldots$ be denoted by

$$f(h), \quad \phi(h),$$

we see that, if

$$f(h) = \phi(a),$$

so that

$$\phi(h) = f(a),$$

then

$$f(h)\phi(h) = f(a)\phi(a);$$

showing that

$$f(h)\phi(h)$$

is an absolute invariant *quâ* the transformation which replaces the elements

$$h_1, h_2, h_3, \ldots$$

by the elements

$$a_1, a_2, a_3, \ldots.$$

Art. 55. With these necessary preliminary remarks I define a new function of weight n, viz. :—

$$h_{p_1p_2p_3\ldots},$$

where $p_1p_2p_3\ldots$ is any composition of the number n; of the given weight there are

$$2^{n-1}$$

such functions, one of which is clearly

$$h_n.$$

The complete definition is given by the multiplication law

$$h_{p_1p_2\ldots p_s}h_{q_1q_2\ldots q_t}$$
$$= h_{p_1p_2\ldots p_sq_1q_2\ldots q_t}+h_{p_1\ldots p_{s-1},\ p_s+q_1,\ q_2\ldots q_t},$$

where the functions

$$h_{p_1p_2\ldots p_s},\quad h_{q_1q_2\ldots q_t}$$

are, or are not, of the same weight.

Art. 56. A second new function

$$a_{p_1p_2p_3\ldots}$$

is similarly defined by the same law; viz.,

$$a_{p_1p_2\ldots p_s}a_{q_1q_2\ldots q_t}$$
$$= a_{p_1\ldots p_sq_1\ldots q_t}+a_{p_1\ldots p_{s-1},\ p_s+q_1,\ q_2\ldots q_t}.$$

What follows applies generally to both of the new functions.

Art. 57. Since the multiplication is commutative, we have the first important property, viz.,

$$h_{p_1p_2\ldots p_sq_1q_2\ldots q_t}+h_{p_1\ldots p_{s-1},\ p_s+q_1,\ q_2\ldots q_t}$$
$$= h_{q_1\ldots q_tp_1\ldots p_s}+h_{q_1\ldots q_{t-1},\ q_t+p_1,\ p_2\ldots p_s}.$$

Art. 58. Every product of elementary functions is expressible in terms of the new functions, *e.g.*,

$$h_ph_q = h_{pq}+h_{p+q},$$
$$h_ph_qh_r = h_{pqr}+h_{p+q,\,r}+h_{p,\,q+r}+h_{p+q+r};$$

and in general

$$h_{p_1}h_{p_2}h_{p_3}\ldots h_{p_s} = \theta_h\{(p_1)(p_2)(p_3)\ldots(p_s)\},$$

where, in $\theta_h\{\quad\}$, the sum of the coefficients is

$$2^{s-1}.$$

These relations show that

$$h_{p_1p_2\ldots p_s} = h_{p_s\ldots p_2p_1}.$$

Art. 59. Similarly

$$h_{pq} = h_ph_q-h_{p+q},$$
$$h_{pqr} = h_ph_qh_r-h_{p+q}h_r-h_ph_{q+r}+h_{p+q+r};$$

and in general

$$h_{p_1p_2\ldots p_s} = \phi_h\{(p_1)(p_2)\ldots(p_s)\}.$$

If, moreover, we define

$$\theta_h\{(p_1p_2\ldots p_s)(q_1q_2\ldots q_t)(r_1r_2\ldots r_u)\}$$

as denoting

$$h_{p_1\dots p_s q_1\dots q_t r_1\dots r_u} + h_{p_1\dots p_{s-1},\, p_s+q_1,\, q_2\dots q_t r_1\dots r_u}$$
$$+ h_{p_1\dots p_s q_1\dots q_{t-1},\, q_t+r_1,\, r_2\dots r_u} + h_{p_1\dots p_{s-1},\, p_s+q_1,\, q_2\dots q_{t-1},\, q_t+r_1,\, r_2\dots r_u},$$

and

$$\phi_h\{(p_1\dots p_{s-1}p_s)\,(q_1q_2\dots q_{t-1}q_t)\,(r_1r_2\dots r_u)\}$$

as denoting

$$h_{p_1\dots p_s} h_{q_1\dots q_t} h_{r_1\dots r_u} - h_{p_1\dots p_{s-1},\, p_s+q_1,\, q_2\dots q_t} h_{r_1\dots r_u}$$
$$- h_{p_1\dots p_s} h_{q_1\dots q_{t-1},\, q_t+r_1,\, r_2\dots r_u} + h_{p_1\dots p_{s-1},\, p_s+q_1,\, q_2\dots q_{t-1},\, q_t+r_1,\, r_2\dots r_u},$$

according to the law usual in this subject; we find

$$\theta_h\{(p_1\dots p_{s-1}p_s)\,(q_1q_2\dots q_{t-1}q_t)\,(r_1r_2\dots r_u)\}$$
$$= h_{p_1\dots p_{s-1}p_s} h_{q_1q_2\dots q_{t-1}q_t} h_{r_1r_2\dots r_u};$$

and

$$\phi_h\{(p_1\dots p_{s-1}p_s)\,(q_1q_2\dots q_{t-1}q_t)\,(r_1r_2\dots r_u)\}$$
$$= \phi_h\{(p_1\dots p_{s-1}\,p_sq_1\dots q_t)\,(r_1\dots r_u)\},$$
$$= \phi_h\{(p_1\dots p_s)\,(q_1\dots q_tr_1\dots r_u)\},$$
$$= \phi_h\{(p_1\dots p_sq_1\dots q_tr_1\dots r_u)\},$$
$$= h_{p_1\dots p_sq_1\dots q_tr_1\dots r_u}.$$

The reader may verify that

$$\phi_h\{(p)\,(q)\,(r)\},\ \phi_h\{(pq)\,(r)\},\ \phi_h\{(p)\,(qr)\},\ \phi_h\{(pqr)\}$$

have each the same value $\quad h_ph_qh_r - h_{p+q}h_r - h_ph_{q+r} + h_{p+q+r},$

which we have denoted by

$$h_{pqr}.$$

Art. 60. I pass on to drag into the light some important relations connecting

$$h_{p_1\dots p_s} \text{ and } a_{p_1\dots p_s}.$$

When the relation

$$h_n = \Sigma(-)^{n+\Sigma k}\frac{(\Sigma k)!}{k_1!\dots k_s!}\,a_1^{k_1}\dots a_s^{k_s}$$

was under observation just now, it will not have escaped notice that this is precisely the expansion of

$$a_{1^n}$$

for

$$a_{11} = a_1^2 - a_2,$$
$$a_{111} = a_1^3 - 2a_1a_2 + a_3;\ \ \&c.,$$

and by the law of formation we see that

$$a_{1^n} = h_n,$$

and thence

$$a_n = h_{1^n}.$$

The known value of h_n is thus given by a law identical with the multiplication law of this paper, and the expression of h_n in terms of

$$a_1,\ a_2,\ a_3,\ \ldots$$

is completely given by

$$h_n = a_{1^n}.$$

This new statement, of a well-known law, immediately suggests the generalization to which I proceed.

Observe that

$$n \text{ and } 1^n$$

are zig-zag conjugate compositions.

From the relation

$$a_{pq} = a_p a_q - a_{p+q}$$

is now deduced

$$a_{pq} = h_{1^p} h_{1^q} - h_{1^{p+q}}\ ;$$

and, since

$$h_{1^p} h_{1^q} = h_{1^{p+q}} + h_{1^{p-1}21^{q-1}},$$

$$a_{pq} = h_{1^{p-1}21^{q-1}}\ ;$$

and we again observe that

$$pq \text{ and } 1^{p-1}21^{q-1}$$

are zig-zag conjugate compositions.

Hence writing

$$(1^{p-1}21^{q-1}) = (pq)',$$

$$a_{(pq)} = h_{(pq)'}\ ;$$

and, in general, I have established (but reserve the proof for another occasion) that

$$a_{(p_1 p_2 \ldots)} = h_{(p_1 p_2 \ldots)'},$$

where

$$(p_1 p_2 \ldots),\ (p_1 p_2 \ldots)'$$

are zig-zag conjugate compositions.

Art. 61. The theorem has an interest of its own, but it is also of vital importance in this investigation. This importance consists partly in the circumstance that the functions

$$h_{pqr\ldots}$$

are those which naturally arise in the present theory of permutations. The present theorem enables the immediate expression of them in terms of the elementary symmetric functions

$$a_1,\ a_2,\ a_3, \ldots$$

and thus they may be more easily dealt with by symmetric functions differential operators. In fact, the homogeneous product sums

$$h_1,\ h_2,\ h_3, \ldots$$

can be made to disappear from the investigation; but, as will be seen, it is sometimes advantageous to retain them wholly or in part.

Art. 62. To gain familiarity with the new functions I give without proof some of their elementary properties.

$$s_n = a_{1^n} - a_{21^{n-2}} + a_{31^{n-3}} - \dots (-)^{n+1} a_n$$
$$= (-)^{n+1} \{h_{1^n} - h_{21^{n-2}} + h_{31^{n-3}} - \dots (-)^{n+1} h_n\},$$

where s_n is the sum of the n^{th} power of the roots.

The following expression for $a_{s1^{n-s}}$

$$a_{s1^{n-s}} = a_{s-1} h_{n-s+1} - a_{s-2} h_{n-s+2} + \dots (-)^{s+1} h_n.$$

The result of operations with D_p, viz.,

$$D_p a_{s1^{n-s}} = a_{s-1} h_{n-s-p+1}.$$

If $s_{n,t}$ denote the sum of the symmetric functions whose partitions contain exactly t parts, we have the companion tables, in which the law is obvious.

	$s_{n,1}$.	$s_{n,2}$.	$s_{n,3}$.	$s_{n,4}$.	$s_{n,5}$.	$s_{n,6}$.
a_{1^n}	1	+1	+1	+1	+1	+ 1
$a_{21^{n-2}}$		1	+2	+3	+4	+ 5
$a_{31^{n-3}}$			1	+3	+6	+10
$a_{41^{n-4}}$				1	+4	+10
$a_{51^{n-5}}$					1	+ 5
$a_{61^{n-6}}$						1

	a_{1^n}.	$a_{21^{n-2}}$.	$a_{31^{n-3}}$.	$a_{41^{n-4}}$.	$a_{51^{n-5}}$.	$a_{61^{n-6}}$.
$s_{n,1}$	1	− 1	+1	− 1	+1	− 1
$s_{n,2}$		1	− 2	+3	− 4	+ 5
$s_{n,3}$			1	− 3	+6	− 10
$s_{n,4}$				1	− 4	+10
$s_{n,5}$					1	− 5
$s_{n,6}$						1

The fundamental properties of these new symmetric functions were communicated by me to Section A of the British Association for the Advancement of Science, at the York meeting, 1906, August 1–8.

Art. 63. The generating function of $N(abc\dots)$ is either

$$h_{abc\dots} \quad \text{or} \quad a_{(abc\dots)'}.$$

We can now determine the highest symmetric function, in dictionary order of the parts, which occurs in the development of $h_{abc\ldots}$. This, by the known theory of symmetric functions, is obtained from the form

$$a_{(abc\ldots)'}$$

by expressing $(abc\ldots)'$ as a partition and taking the Ferrers conjugate $(abc\ldots)''$; then we see that no symmetric function, prior in dictionary order to

$$(abc\ldots)'',$$

can appear.

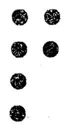
Also the highest integer in $$(abc\ldots)'$$

is the lower limit of the number of parts, occurring in the partition of a symmetric function, arising from the development of

$$h_{abc}\ldots.$$

E.g., since $$h_{141} = a_{21^22},$$

we arrange 21^22 as a partition, obtaining 2^21^2, and taking the Ferrers conjugate from the graph



we reach (42) as the highest symmetric function in dictionary order that occurs in the development of h_{141}.

Hence $$\text{N}\,(141)_6 = \text{N}\,(141)_{51} = 0.$$

(See the table of weight 6.)

Numerous relations such as

$$h_{141}+h_{51} = h_{41^2}+h_{42}$$

can be verified by the same table.

Art. 64. Before proceeding to establish the multiplication theorem, the generalization of that in Part I., it is necessary to examine the mode of operation of the differential operator $$\text{D}_a$$

upon a product $$h_{p_1}{}^{\pi_1}h_{p_2}{}^{\pi_2}\ldots,$$

or $$a_{p_1}{}^{\pi_1}a_{p_2}{}^{\pi_2}\ldots.$$

It is clear that $$\text{D}_a h_p = h_{p-a}.$$

In the paper it was shown that

$$\text{D}_a h_p h_q = \Sigma\,(\text{D}_{a'}h_p)(\text{D}_{a''}h_q),$$

where $a'a''$ denotes a composition of a into two parts, zero not excluded, and the summation is for every such composition.

Hence
$$D_a h_p h_q = \Sigma h_{p-a'} h_{q-a''}.$$

E.g.,
$$D_4 h_4 h_3 = h_3 + h_1 h_2 + h_3 + h_2 h_1 = 2h_3 + 2h_2 h_1,$$

where the compositions of 4 have been taken in the order

40, 31, 13, 22.

In general
$$D_a h_{p_1} h_{p_2} \ldots h_{p_s} = \Sigma h_{p_1 - a'} h_{p_2 - a''} \ldots h_{p_s - a^{(s)}},$$

where
$$a' a'' \ldots a^{(s)}$$

is a composition of a into s or fewer parts.

It is to be noted that in forming the compositions zeros are parts, so that, for instance,

400, 040, 004

count as different compositions.

If the operand be
$$a_{p_1} a_{p_2} \ldots a_{p_s},$$

since
$$D_a (a_{p_1}) = 0 \text{ unless } a = 1,$$

we need only attend to the compositions composed of units and zeros.

Thus
$$D_2 a_4 a_5 a_6 a_7 = a_3 a_4 a_6 a_7 + a_3 a_5^2 a_7 + a_3 a_5 a_6^2 + a_4^2 a_5 a_7 + a_4^2 a_6^2 + a_4 a_5^2 a_6.$$

It is easy to show that
$$D_1 h_{ab} = h_{a-1,\,b} + h_{a,\,b-1} + h_{a+b-1},$$

from which
$$N\,(ab)_{pqr\ldots 1} = N\,(a-1,\ b)_{pqr\ldots} + N\,(a,\ b-1)_{pqr\ldots} + N\,(a+b-1)_{pqr\ldots};$$

and, particularly,
$$N\,(42)_{2^2 1^2} = N\,(32)_{2^2 1} + N\,(41)_{2^2 1} + N\,(5)_{2^2 1}$$
$$7 \quad = \quad 4 \quad + \quad 2 \quad + \quad 1$$

from the table.

Similar formulæ can be established at pleasure.

The Conjugate Law.

Art. 65. It has been seen (Art. 6) that, when the numbers permuted are specified by
$$1^n,$$
$$N\,(pq\ldots) = N\,(pq\ldots)',$$

where
$$(pq\ldots),\quad (pq\ldots)'$$

denote conjugate compositions.

We write the theorem
$$N\,(pq\ldots)_{(1^n)} = N\,(pq\ldots)'_{(1^n)};$$

and we may inquire into the existence of an analogous theorem when the numbers permuted have any other specification.

Consider the expression

$$h_{(pq\ldots)} - h_{(pq\ldots)'},$$

which is the generating function for the difference between

$$\mathrm{N}\,(pq\ldots) \quad \text{and} \quad \mathrm{N}\,(pq\ldots)',$$

for all specifications of the numbers permuted.

The generating function may be written

$$h_{pq\ldots} - a_{pq\ldots},$$

according to the theorem proved above.

The differential operation

$$\mathrm{D}_1$$

has the equivalent forms

$$\partial_{a_1} + a_1\,\partial_{a_2} + a_2\,\partial_{a_3} + \ldots$$

$$\partial_{h_1} + h_1\,\partial_{h_2} + h_2\,\partial_{h_3} + \ldots;$$

hence

$$\mathrm{D}_1^{\mu} h_{pq\ldots}$$

is the same function of

$$h_1,\ h_2,\ h_3,\ \ldots$$

that

$$\mathrm{D}_1^{\mu} a_{pq\ldots}$$

is of

$$a_1,\ a_2,\ a_3,\ \ldots.$$

It follows at once that

$$\mathrm{D}_1^{n}\,(h_{pq\ldots} - a_{pq\ldots}) = 0,$$

equivalent to the known result

$$\mathrm{N}\,(pq\ldots)_{(1^n)} = \mathrm{N}\,(pq\ldots)'_{(1^n)}$$

already found.

Art. 66. Now, considering the generating functions

$$h_{(pq\ldots)} + h_{(pq\ldots)'}$$

or

$$h_{(pq\ldots)} + a_{(pq\ldots)},$$

$$\mathrm{D}_1^{n-2}\,\{h_{pq\ldots} + a_{pq\ldots}\}$$

must be of the form

$$\mathrm{A}h_2 + \mathrm{B}h_1^2 + \mathrm{A}a_2 + \mathrm{B}a_1^2,$$

or

$$(\mathrm{A} + 2\mathrm{B})\,\{(2) + 2\,(1^2)\}.$$

Hence

$$\mathrm{D}_1^{n-2}\mathrm{D}_2\,(h_{pq\ldots} + a_{pq\ldots})$$

$$= \tfrac{1}{2}\mathrm{D}_1^{n}\,(h_{pq\ldots} + a_{pq\ldots})$$

$$= \mathrm{D}_1^{n} h_{pq\ldots},$$

equivalent to

$$\mathrm{N}\,(pq\ldots)_{(21^{n-2})} + \mathrm{N}\,(pq\ldots)'_{(21^{n-2})} = \mathrm{N}\,(pq\ldots)_{(1^n)}.$$

Thus, from the table $n = 6$,

$$\mathrm{N}\,(33)_{(21^4)} + \mathrm{N}\,(1^2 21^2)_{(21^4)} = \mathrm{N}\,(33)_{(1^6)}$$

$$13 \quad + \quad 6 \quad = \quad 19.$$

Art. 67. Again, operating with

$$D_1^{n-3},$$

upon

$$h_{pq\ldots}-a_{pq\ldots},$$

we obtain a result of the form

$$A(h_3-a_3)+B(h_2h_1-a_2a_1)$$

or

$$(A+B)\{(3)+(21)\};$$

hence

$$D_1^{n-3}D_3(h_{pq\ldots}-a_{pq\ldots}) = D_1^{n-2}D_2(h_{pq\ldots}-a_{pq\ldots}),$$

equivalent to

$$N(pq\ldots)_{(31^{n-3})}-N(pq\ldots)'_{(31^{n-3})} = N(pq\ldots)_{(21^{n-2})}-N(pq\ldots)'_{(21^{n-2})};$$

and, particularly, from the table

$$N(321)_{(31^3)}-N(2^21^2)_{(31^3)} = N(321)_{(21^4)}-N(2^21^2)_{(21^4)}$$
$$8 - 3 = 20 - 15.$$

No new result is obtained by taking

$$h_{pq\ldots}+a_{pq\ldots}$$

as the operand.

Art. 68. Further,

$$D_1^{n-4}(h_{pq\ldots}-a_{pq\ldots})$$

has the form

$$A(h_4-a_4)+B(h_3h_1-a_3a_1)+C(h_2^2-a_2^2)+D(h_2h_1^2-a_2a_1^2),$$

reducing to

$$(A+B+C+D)(4)+(A+2B+2C+2D)\{(31)+(2^2)+(21^2)\},$$

equivalent to the new result

$$N(pq\ldots)_{(2^21^{n-4})}-N(pq\ldots)'_{(2^21^{n-4})} = N(pq\ldots)_{(31^{n-3})}-N(pq\ldots)'_{(31^{n-3})};$$

and, particularly, from the table

$$N(2^3)_{(2^21^2)}-N(12^21)_{(2^21^2)} = N(2^3)_{(31^3)}-N(12^21)_{(31^3)}$$
$$18 - 13 = 11 - 6.$$

Art. 69. If we take here the operand to be

$$h_{pq\ldots}+a_{pq\ldots},$$

a new result is obtained, viz.,

$$2N(pq\ldots)_{(41^{n-4})}+2N(pq\ldots)'_{(41^{n-4})}+N(pq\ldots)_{(21^{n-2})}+N(pq\ldots)'_{(21^{n-2})}$$
$$= 2N(pq\ldots)_{(31^{n-3})}+2N(pq\ldots)'_{(31^{n-3})}+N(pq\ldots)_{2^21^{n-4}}+N(pq\ldots)'_{(2^21^{n-4})}.$$

The above is sufficient to indicate the nature of the results which present themselves; I have not attempted to generalise them. The question appears to be a difficult one.

Section 7.

Generalisation of the Multiplication Theorem.

Art. 70. I will establish the result

$$\Sigma\Sigma\Sigma\ldots\{\mathrm{N}(a_1a_2a_3\ldots)_{(p_1q_1r_1\ldots)}\mathrm{N}(b_1b_2b_3\ldots)_{(p_2q_2r_2\ldots)}\mathrm{N}(c_1c_2c_3\ldots)_{(p_3q_3r_3\ldots)}\ldots\}$$
$$= \theta_{\mathrm{N}}\{(a_1a_2a_3\ldots)(b_1b_2b_3\ldots)(c_1c_2c_3\ldots)\ldots\}_{(pqr\ldots)},$$

where the summation is for all solutions of the diophantine equations

$$p_1+q_1+r_1+\ldots = \Sigma a,$$
$$p_2+q_2+r_2+\ldots = \Sigma b,$$
$$p_3+q_3+r_3+\ldots = \Sigma c,$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$p_1+p_2+p_3+\ldots = p,$$
$$q_1+q_2+q_3+\ldots = q,$$
$$r_1+r_2+r_3+\ldots = r,$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

For consider

$$\theta_{\mathrm{N}}\{(a_1a_2)(b_1b_2)\}$$
$$= \phi_{\mathrm{C}}\{(a_1)(a_2b_1)(b_2)\},$$
$$= \mathrm{C}(a_1a_2b_1b_2)_{(pqr\ldots)} - \mathrm{C}(a_1+a_2,\, b_1b_2)_{(pqr\ldots)} - \mathrm{C}(a_1a_2,\, b_1+b_2)_{(pqr\ldots)} + \mathrm{C}(a_1+a_2,\, b_1+b_2)_{(pqr\ldots)}$$
$$= (\mathrm{D}_{a_1}\mathrm{D}_{a_2}\mathrm{D}_{b_1}\mathrm{D}_{b_2} - \mathrm{D}_{a_1+a_2}\mathrm{D}_{b_1}\mathrm{D}_{b_2} - \mathrm{D}_{a_1}\mathrm{D}_{a_2}\mathrm{D}_{b_1+b_2} + \mathrm{D}_{a_1+a_2}\mathrm{D}_{b_1+b_2})h_ph_qh_r\ldots$$
$$= (\mathrm{D}_{a_1}\mathrm{D}_{a_2} - \mathrm{D}_{a_1+a_2})(\mathrm{D}_{b_1}\mathrm{D}_{b_2} - \mathrm{D}_{b_1+b_2})h_ph_qh_r\ldots$$
$$= \mathrm{D}_{a_1a_2}\mathrm{D}_{b_1b_2}h_ph_qh_r\ldots = \mathrm{D}_p\mathrm{D}_q\mathrm{D}_r\ldots h_{a_1a_2}h_{b_1b_2}.$$

Now

$$\mathrm{D}_p\mathrm{D}_q\mathrm{D}_r\ldots h_{a_1a_2}h_{b_1b_2}$$
$$= \Sigma\Sigma\Sigma\ldots(\mathrm{D}_{p_1}\mathrm{D}_{q_1}\mathrm{D}_{r_1}\ldots h_{a_1a_2})(\mathrm{D}_{p_2}\mathrm{D}_{q_2}\mathrm{D}_{r_2}\ldots h_{b_1b_2}),$$

the summation being for all solutions of the diophantine equations

$$p_1+q_1+r_1+\ldots = a_1+a_2,$$
$$p_2+q_2+r_2+\ldots = b_1+b_2,$$
$$p_1+p_2 = p,$$
$$q_1+q_2 = q,$$
$$r_1+r_2 = r,$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot$$

Moreover,

$$\mathrm{D}_{p_1}\mathrm{D}_{q_1}\mathrm{D}_{r_1}\ldots h_{a_1a_2} = \mathrm{N}(a_1a_2)_{p_1q_1r_1\ldots}.$$

Hence

$$\theta_{\mathrm{N}}\{(a_1a_2)(b_1b_2)\}_{(pqr\ldots)}$$
$$= \Sigma\,\mathrm{N}(a_1a_2)_{p_1q_1r_1\ldots}\mathrm{N}(b_1b_2)_{p_2q_2r_2\ldots};$$

and, by like reasoning, the theorem as enunciated follows.

As examples,
$$N(321)_{(222)}+N(33)_{(222)} = 3N(32)_{(221)},$$
derived from
$$\theta_N\{(32)(1)\}_{(222)};$$
and
$$N(24)_{(222)}+N(231)_{(222)}+N(213)_{(222)}+N(222)_{(222)}$$
$$+N(2211)_{(222)}+N(2121)_{(222)}+N(2112)_{(222)}+N(21111)_{(222)}$$
$$= 6N(21)_{(111)}+18N(21)_{(21)},$$
derived from
$$\theta_N\{(21)(1)^3\}_{(222)},$$

Art. 71. The enumeration of the permutations, whose specifications contain a given largest integer, will now be investigated.

Let
$$I_m,\ J_m,\ K_m$$
denote respectively
$$\Sigma N(abc\ldots)$$
in which

(i.) the highest of the integers a, b, c, ... is m or less;
(ii.) " " " " or greater;
(iii.) " " " " exactly;

so that, when $a+b+c+\ldots = n$,
$$I_m = K_1+K_2+\ldots+K_m,$$
$$J_m = K_m+K_{m+1}+\ldots+K_m,$$
$$I_m = J_1 = I_m+J_m-K_m,$$
$$I_m-I_{m-1} = J_m-J_{m+1} = K_m,$$
$$\theta_N(n) = N(n) = J_n = K_n = I_n-I_{n-1};$$
and, since
$$\theta_N\{(n-1)(1)\} = N(n-1,\ 1)+N(n),$$
$$\theta_N\{(1)(n-1)\} = N(1,\ n-1)+N(n),$$
we find
$$2\theta_N\{(n-1)(1)\}$$
$$= K_{n-1}+2K_n = J_{n-1}+J_n = -I_{n-1}-I_{n-2}+2I_n;$$
also
$$\theta_N\{(n-2)(1)^2\} = N(n-2,\ 1^2)+N(n-1,\ 1)+N(n-2,\ 2)+N(n),$$
$$\theta_N\{(1)(n-2)(1)\} = N(1,\ n-2,\ 1)+N(n-1,\ 1)+N(1,\ n-1)+N(n),$$
$$\theta_N\{(1)^2(n-2)\} = N(1^2,\ n-2)+N(1,\ n-1)+N(2n-2)+N(n),$$
and by addition
$$3\theta_N\{(n-2)(1)^2\} = K_{n-2}+2K_{n-1}+3K_n,$$
$$= J_{n-2}+J_{n-1}+J_n,$$
$$= -I_{n-3}-I_{n-2}-I_{n-1}+3I_n,$$
the law apparent here obtains so long as a number $n-\nu$ appearing in
$$N(\quad),$$

on the right-hand side, is not equal to any other number in the same bracket; so that, *when* $s<\frac{1}{2}n$,

$$(s+1)\,\theta_{\mathrm{N}}\{(n-s)\,(1)^s\}$$
$$= \mathrm{K}_{n-s}+2\mathrm{K}_{n-s+1}+3\mathrm{K}_{n-s+2}+\ldots+(s+1)\,\mathrm{K}_n,$$
$$= \mathrm{J}_{n-s}+\mathrm{J}_{n-s+1}+\mathrm{J}_{n-s+2}+\ldots+\mathrm{J}_n,$$
$$= -\mathrm{I}_{n-s-1}-\mathrm{I}_{n-s}-\mathrm{I}_{n-s+1}-\ldots+(s+1)\,\mathrm{I}_n.$$

Hence

$$\mathrm{J}_m = (n-m+1)\,\theta_{\mathrm{N}}\{(m)\,(1)^{n-m}\}-(n-m)\,\theta_{\mathrm{N}}\{(m+1)\,(1)^{n-m-1}\},$$
$$\mathrm{K}_m = (n-m+1)\,\theta_{\mathrm{N}}\{(m)\,(1)^{n-m}\}-2\,(n-m)\,\theta_{\mathrm{N}}\{(m+1)\,(1)^{n-m-1}\}+(n-m-1)\,\theta_{\mathrm{N}}\{(m+2)\,(1)^{n-m-2}\},$$

and, the specification of the numbers permuted being

$$(pqr\ldots),$$

$$\mathrm{I}_m = \frac{n!}{p!q!r!\ldots}-(n-m)\,\theta_{\mathrm{N}}\{(m+1)\,(1)^{n-m-1}\}+(n-m-1)\,\theta_{\mathrm{N}}\{(m+2)\,(1)^{n-m-2}\}.$$

Now

$$\theta_{\mathrm{N}}\{(a)\,(b)\,(c)\,\ldots\} = \mathrm{C}\,(abc\ldots) = \mathrm{D}_a\mathrm{D}_b\mathrm{D}_c\ldots h_ph_qh_r\ldots\,;$$

thence

$$\mathrm{J}_m$$
$$= \{(n-m+1)\,\mathrm{D}_m\mathrm{D}_1^{\,n-m}-(n-m)\,\mathrm{D}_{m+1}\mathrm{D}_1^{\,n-m-1}\}\,h_ph_qh_r\ldots,$$
$$= \mathrm{D}_p\mathrm{D}_q\mathrm{D}_r\ldots\{(n-m+1)\,h_mh_1^{\,n-m}-(n-m)\,h_{m+1}h_1^{\,n-m-1}\}\,;$$

or, m not being less than the greatest integer in $\frac{1}{2}(n+1)$,

$$(n-m+1)\,h_mh_1^{\,n-m}-(n-m)\,h_{m+1}h_1^{\,n-m-1}$$

is the generating function of the number J_m.

Similarly

$$\mathrm{K}_m$$
$$= \mathrm{D}_p\mathrm{D}_q\mathrm{D}_r\ldots\{(n-m+1)\,h_mh_1^{\,n-m}-2\,(n-m)\,h_{m+1}h_1^{\,n-m-1}+(n-m-1)\,h_{m+2}h_1^{\,n-m-2}\}\,;$$

and, m not being less than the greatest integer in $\frac{1}{2}(n+1)$,

$$(n-m+1)\,h_mh_1^{\,n-m}-2\,(n-m)\,h_{m+1}h_1^{\,n-m-1}+(n-m-1)\,h_{m+2}h_1^{\,n-m-2}$$

is the generating function of the number K_m.

Similarly, but subject now to the condition that m must not be less than the greatest integer in $\frac{1}{2}(n-1)$,

$$(n-m-1)\,h_{m+2}h_1^{\,n-m-2}-(n-m)\,h_{m+1}h_1^{\,n-m-1}$$

is the function which generates the number

$$\mathrm{I}_m-\frac{n!}{p!q!r!\ldots}.$$

Subject to the conditions mentioned, we have a complete solution of the problem, but when m has other values, the solution is less simple and I see no way of effecting it.

Section 8.

Art. 72. I recall that the number of ways of distributing numbers (or objects) specified by

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots),$$

into m different parcels, is given by the series

$$\begin{aligned} F_m = & \binom{m+p_1-1}{p_1}^{\pi_1} \binom{m+p_2-1}{p_2}^{\pi_2} \binom{m+p_3-1}{p_3}^{\pi_3} \ldots \\ & - \binom{m}{1} \binom{m+p_1-2}{p_1}^{\pi_1} \binom{m+p_2-2}{p_2}^{\pi_2} \binom{m+p_3-2}{p_3}^{\pi_3} \ldots \\ & + \binom{m}{2} \binom{m+p_1-3}{p_1}^{\pi_1} \binom{m+p_2-3}{p_2}^{\pi_2} \binom{m+p_3-3}{p_3}^{\pi_3} \ldots \\ & - \ldots ; \end{aligned}$$

and this, for brevity, I write

$$F_m = G_m - \binom{m}{1} G_{m-1} + \binom{m}{2} G_{m-2} - \ldots .$$

Let

$$N_{m, p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots}$$

denote the number of distributions, associated with a descending specification containing exactly m parts, and write this

$$N_m,$$

when there is no risk of misunderstanding.

Following the proof of Art. 23, it may be proved that

$$F_m = N_m + \binom{n-m+1}{1} N_{m-1} + \binom{n-m+2}{2} N_{m-2} + \ldots + \binom{n-1}{m-1} N_1 ;$$

and also

$$N_m = F_m - \binom{n-m+1}{1} F_{m-1} + \binom{n-m+2}{2} F_{m-2} + \ldots + (-)^{m+1} \binom{n-1}{m-1} F_1 ;$$

and thence

$$N_m = G_m - \binom{n+1}{1} G_{m-1} + \binom{n+1}{2} G_{m-2} - \ldots + (-)^{m+1} \binom{n+1}{m-1} G_1.$$

Art. 73. From this relation the following results are obtained :—

$n = 3.$

	(3).	(21).	(1^3).
N_1	1	1	1
N_2		2	4
N_3			1

$n = 4.$

	(4).	(31).	(2^2).	(21^2).	(1^4).
N_1	1	1	1	1	1
N_2		3	4	7	11
N_3			1	4	11
N_4					1

$n = 5.$

	(5).	(41).	(32).	(31^2).	(2^21).	(21^3).	(1^5).
N_1	1	1	1	1	1	1	1
N_2		4	6	10	12	18	26
N_3			3	9	15	33	66
N_4					2	8	26
N_5							1

$n = 6.$

	(6).	(51).	(42).	(41^2).	(3^2).	(321).	(31^3).	(2^3).	(2^21^2).	(21^4).	(1^6).
N_1	1	1	1	1	1	1	1	1	1	1	1
N_2		5	8	13	9	17	25	20	29	41	57
N_3			6	16	9	33	67	48	93	171	302
N_4					1	9	27	20	53	131	302
N_5								1	4	16	57
N_6											1

To explain, observe that the number at the intersection of the row N_3 and the column (2^21^2) shows that
$$N_{3, 2^21^2} = 93.$$

These tables will be of constant service in verifying results to be obtained.

Art. 74. From the relation

$$N_m = G_m - \binom{n+1}{1} G_{m-1} + \binom{n+1}{2} G_{m-2} - \ldots,$$

we can obtain a system, for, summing each side from $m = 1$ to $m = m$,

$$N_m + N_{m-1} + \ldots + N_1 = G_m - \binom{n}{1} G_{m-1} + \binom{n}{2} G_{m-2} - \ldots,$$

and, repeating the summation θ times,

$$N_m+\binom{\theta+1}{1}N_{m-1}+\binom{\theta+2}{2}N_{m-2}+\ldots+\binom{\theta+m-1}{m-1}N_1 = G_m-\binom{n-\theta}{1}G_{m-1}+\binom{n-\theta}{2}G_{m-2}-\ldots;$$

so that, when $\theta = n$,

$$N_m+\binom{n+1}{1}N_{m-1}+\ldots+\binom{n+m-1}{m-1}N_1 = G_m.$$

Again, taking differences instead of summing, we get the series

$$N_m-N_{m-1} = G_m-\binom{n+2}{1}G_{m-1}+\binom{n+2}{2}G_{m-2}-\ldots,$$

$$N_m-2N_{m-1}+N_{m-2} = G_m-\binom{n+3}{1}G_{m-1}+\binom{n+3}{2}G_{m-2}-\ldots,$$

and in general

$$N_m-\binom{p}{1}N_{m-1}+\ldots\pm N_{m-p}$$

$$= G_m-\binom{n+p+1}{1}G_{m-1}+\binom{n+p+1}{2}G_{m-2}-\ldots.$$

These results are all given by the two formulæ

$$N_m+\binom{p}{1}N_{m-1}+\binom{p+1}{2}N_{m-2}+\ldots$$

$$= G_m-\binom{n+1-p}{1}G_{m-1}+\binom{n+1-p}{2}G_{m-2}-\ldots;$$

$$N_m-\binom{p}{1}N_{m-1}+\binom{p}{2}N_{m-2}-\ldots$$

$$= G_m-\binom{n+1+p}{1}G_{m-1}+\binom{n+1+p}{2}G_{m-2}-\ldots;$$

which become the same when $p = 0$.

Curious Expression for N_m.

Art. 75. I shall now prove that

$$N_m = P_{m-1}-\frac{m-2}{m-1}\binom{n}{1}P_{m-2}+\frac{m-3}{m-1}\binom{n}{2}P_{m-3}+\ldots+(-)^m\frac{1}{m-1}\binom{n}{m-2}P_1,$$

where

$$P_s = \frac{\binom{p_1+s-1}{p_1}^{\pi_1}\binom{p_2+s-1}{p_2}^{\pi_2}\ldots}{s^{\pi_1+\pi_2+\ldots}}\{(p_1+s)^{\pi_1}(p_2+s)^{\pi_2}\ldots\}_{-1,0},$$

where

$$\{(p_1+s)^{\pi_1}(p_2+s)^{\pi_2}\ldots\}_{-1,0}$$

denotes the expansion of

$$(p_1+s)^{\pi_1}(p_2+s)^{\pi_2}\ldots,$$

when deprived of the terms linear in p_1, p_2, ..., and of the term independent of $p_1, p_2, \ldots$.

For it is easy to show that two consecutive terms

$$(-)^t \frac{m-t-1}{m-1}\binom{n}{t} \mathrm{P}_{m-t-1} + (-)^{t+1}\frac{m-t-2}{m-1}\binom{n}{t+1}\mathrm{P}_{m-t-2}$$

may be given the form

$$(-)^t \frac{m-t-1}{m-1}\binom{n}{t}\binom{p_1+m-t-1}{p_1}^{\pi_1}\binom{p_2+m-t-1}{p_2}^{\pi_2}\ldots$$

$$+(-)^{t+1}\binom{n+1}{t+1}\binom{p_1+m-t-2}{p_1}^{\pi_1}\binom{p_2+m-t-2}{p_2}^{\pi_2}\ldots$$

$$+(-)^t \frac{n+m-t-2}{m-1}\binom{n}{t+1}\binom{p_1+m-t-3}{p_1}^{\pi_1}\binom{p_2+m-t-3}{p_2}^{\pi_2}\ldots,$$

and, giving t the values 0, 2, 4, ..., and summing and simplifying, we obtain

$$\binom{p_1+m-1}{p_1}^{\pi_1}\binom{p_2+m-1}{p_2}^{\pi_2}\ldots$$

$$-\binom{n+1}{1}\binom{p_1+m-2}{p_1}^{\pi_1}\binom{p_2+m-2}{p_2}^{\pi_2}\ldots$$

$$+\binom{n+1}{2}\binom{p_1+m-3}{p_1}^{\pi_1}\binom{p_2+m-3}{p_2}^{\pi_2}\ldots$$

$$-\ldots,$$

which we know to be the value of N_m.

Art. 76. The symmetry of the numbers N_{m,p^π} will not escape the notice of the reader.

SECTION 9.

Art. 77. My purpose now is to connect the preceding pages with my Memoir on the Compositions of Numbers, to which attention has already been directed. In the course of that investigation I had occasion to consider the permutations of the letters in

$$\alpha^p\beta^q\gamma^r,$$

with the object of determining the number of permutations containing given numbers of

$\beta\alpha$ contacts,

$\gamma\alpha$,,

$\gamma\beta$,,

If we take any permutation

$$\ldots\beta\alpha\ldots\gamma\alpha\ldots\gamma\beta\ldots\gamma\beta\alpha\ldots$$

and particularly notice all of such contacts, it is clear that the numbers of parts in the descending specification α, β, γ, ..., being numbers in descending order of magnitude, is necessarily one greater than the number of such contacts; in the present instance there are 6 parts in the descending specification and 5 contacts. The problem of the determination of the permutations having descending specifications containing m parts is identical with that which is concerned with those having $m-1$ contacts of the nature specified.

Art. 78. I established in the Memoir that the letters in

$$\alpha^p\beta^q\gamma^r$$

can be permuted in

$$\binom{s_{21}+s_{31}}{s_{21}}\binom{p}{s_{21}+s_{31}}\binom{q}{s_{32}}\binom{q+s_{31}}{s_{21}+s_{31}}\binom{r}{s_{31}+s_{32}}$$

ways so as to have exactly

s_{21} $\beta\alpha$ contacts,

s_{31} $\gamma\alpha$,,

s_{32} $\gamma\beta$,,

and I further discovered that this number is the coefficient of

$$\lambda_{21}{}^{s_{21}}\lambda_{31}{}^{s_{31}}\lambda_{32}{}^{s_{32}}\alpha^p\beta^q\gamma^r$$

in the development of the function

$$(\alpha+\lambda_{21}\beta+\lambda_{31}\gamma)^p\,(\alpha+\beta+\lambda_{32}\gamma)^q\,(\alpha+\beta+\gamma)^r.$$

Art. 79. In the same paper I showed that for this function may be substituted the function

$$\frac{1}{1-(\alpha+\beta+\gamma)+(1-\lambda_{21})\,\alpha\beta+(1-\lambda_{31})\,\alpha\gamma+(1-\lambda_{32})\,\beta\gamma-(1-\lambda_{21})\,(1-\lambda_{32})\,\alpha\beta\gamma},$$

which does not involve p, q, r, and may therefore be regarded as the general generating function of the numbers.

Art. 80. Reserving for the present the generalizations, which were also given in the papers referred to, it is clear that the application to the present question is obtained by putting

$$\lambda_{21}=\lambda_{31}=\lambda_{32}=\lambda,$$

when we find that the number of permutations of

$$\alpha^p\beta^q\gamma^r,$$

which have descending specifications containing m parts, is the coefficient of

$$\lambda^{m-1}\alpha^p\beta^q\gamma^r$$

in the development of

$$(\alpha+\lambda\beta+\lambda\gamma)^p\,(\alpha+\beta+\lambda\gamma)^q\,(\alpha+\beta+\gamma)^r,$$

or of

$$\frac{1}{1-(\alpha+\beta+\gamma)+(1-\lambda)\,(\alpha\beta+\alpha\gamma+\beta\gamma)-(1-\lambda)^2\alpha\beta\gamma}.$$

This, therefore, is the true generating function of the numbers N_m.

It may be verified, for example, that the complete coefficient of

$$\alpha^2\beta^2\gamma^2$$

is

$$(1+20\lambda+48\lambda^2+20\lambda^3+\lambda^4),$$

which agrees with a previous result.

From a previous result also the coefficient of

$$\lambda^{m-1}\alpha^p\beta^q\gamma^r$$

is

$$\binom{m+p-1}{p}\binom{m+q-1}{q}\binom{m+r-1}{r}$$
$$-\binom{n+1}{1}\binom{m+p-2}{p}\binom{m+q-2}{q}\binom{m+r-2}{r}$$
$$+\binom{n+1}{2}\binom{m+p-3}{p}\binom{m+q-3}{q}\binom{m+r-3}{r}$$
$$-\ldots,$$

where $n=p+q+r$.

Art. 81. Observe that the generating function is a symmetric function of α, β, γ, verifying a previous conclusion that an N_m number is not altered by any interchange of the letters α, β, γ.

When the numbers p, q, r are equal, that is when the objects are specified by the partition

$$(p^3),$$

we can establish a symmetrical property of the numbers N.

For coefficient

$$\lambda^{m-1}(\alpha\beta\gamma)^p \text{ in } (\alpha+\lambda\beta+\lambda\gamma)^p\,(\alpha+\beta+\lambda\gamma)^p\,(\alpha+\beta+\gamma)^p$$

is, by writing

$$\frac{1}{\lambda} \text{ for } \lambda \text{ and } \lambda\alpha,\ \lambda\beta,\ \lambda\gamma \text{ for } \alpha,\ \beta,\ \gamma,$$

equal to coefficient of

$$\lambda^{3p-m+1}(\alpha\beta\gamma)^p \text{ in } (\lambda\alpha+\beta+\gamma)^p\,(\lambda\alpha+\lambda\beta+\gamma)^p\,(\lambda\alpha+\lambda\beta+\lambda\gamma)^p,$$

equal to coefficient of

$$\lambda^{2p-m+1}(\alpha\beta\gamma)^p \text{ in } (\lambda\alpha+\beta+\gamma)^p\,(\lambda\alpha+\lambda\beta+\gamma)^p\,(\alpha+\beta+\gamma)^p,$$

equal to coefficient of

$$\lambda^{2p-m+1}(\alpha\beta\gamma)^p \text{ in } (\alpha+\lambda\beta+\lambda\gamma)^p\,(\alpha+\beta+\lambda\gamma)^p\,(\alpha+\beta+\gamma)^p.$$

Art. 82. Hence

$$N_m = N_{2p-m+2},$$

and the numbers N range from

$$N_1 \text{ to } N_{2p+1},$$

showing that $2p+1$ is the maximum number of parts in the descending specification, when the objects are specified by the partition

$$(p^3).$$

Art. 83. In general, when there are k *different* letters,

$$\alpha_1, \alpha_2, \ldots, \alpha_k,$$

the number of permutations of

$$\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_k^{p_k},$$

which have descending specifications containing m parts, is the coefficient of

$$\lambda^{m-1}\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_k^{p_k}$$

in the development of

$$\{\alpha_1+\lambda(\alpha_2+\ldots+\alpha_k)\}^{p_1}\{\alpha_1+\alpha_2+\lambda(\alpha_3+\ldots+\alpha_k)\}^{p_2}\ldots\{\alpha_1+\alpha_2+\ldots+\lambda\alpha_k\}^{p_{k-1}}\{\alpha_1+\alpha_2+\ldots+\alpha_k\}^{p_k};$$

or of

$$\frac{1}{1-\Sigma\alpha_1+(1-\lambda)\Sigma\alpha_1\alpha_2-(1-\lambda)^2\Sigma\alpha_1\alpha_2\alpha_3+\ldots+(-)^k(1-\lambda)^{k-1}\alpha_1\alpha_2\alpha_3\ldots\alpha_k}.$$

This is the general generating function of the numbers

$$N_m.$$

Art. 84. Since it is symmetrical in regard to

$$\alpha_1, \alpha_2, \ldots, \alpha_k,$$

the value of N_m is not affected by permutation of the letters

$$\alpha_1, \alpha_2, \ldots, \alpha_k.$$

Art. 85. It can be shown also, as in the simpler case, that when

$$p_1 = p_2 = \ldots = p_k = p,$$

the coefficient of

$$\lambda^{m-1}(\alpha_1\alpha_2\ldots\alpha_k)^p$$

is equal to the coefficient of

$$\lambda^{(k-1)p-m+1}(\alpha_1\alpha_2\ldots\alpha_k)^p;$$

so that

$$N_m = N_{(k-1)p-m+2};$$

the numbers N range from

$$N_1 \text{ to } N_{(k-1)p+1};$$

and $(k-1)p+1$ is the maximum number of parts in a descending specification.

Section 10.

Art. 86. The generating function

$$\frac{1}{1-\Sigma\alpha_1+(1-\lambda)\Sigma\alpha_1\alpha_2-(1-\lambda)^2\Sigma\alpha_1\alpha_2\alpha_3+\ldots+(-)^k(1-\lambda)^{k-1}\alpha_1\alpha_2\ldots\alpha_k}$$

now presents itself for examination.

Introducing the elementary functions

$$a_1, a_2, \ldots,$$

and writing $1-\lambda = b$, it is written

$$\frac{1}{1-a_1+ba_2-b^2a_3+\ldots+(-)^k b^{k-1}a_k};$$

or

$$\frac{1}{1-A},$$

where

$$A = a_1-ba_2+b^2a_3-\ldots+(-)^{k+1}b^{k-1}a_k.$$

For the present purpose we may consider k to be infinite, and write

$$A = a_1-ba_2+b^2a_3-\ldots.$$

Art. 87. Taking the symmetric function operators

$$d_s = \partial_{a_s}+a_1\partial_{a_{s+1}}+a_2\partial_{a_{s+2}}+\ldots,$$

$$D_s = \frac{1}{s!}(\partial_{a_1}+a_1\partial_{a_2}+a_2\partial_{a_3}+\ldots)^s = \frac{1}{s!}(d_1^s),$$

and an auxiliary fictitious equation

$$x^r-D_1x^{r-1}+D_2x^{r-2}-D_3x^{r-3}+\ldots = 0,$$

r being an infinite number, it is necessary to remind the reader of the relations existing between the operators.

Successive linear operations of $d_\lambda,\ d_\mu,\ d_\nu,\ \ldots$

are denoted by placing them in separate brackets, thus,

$$(d_\lambda)(d_\mu)(d_\nu)\ldots,$$

but when they are multiplied, as in TAYLOR's theorem, so as to produce a single operator of higher order, they will be placed in one bracket, thus,

$$(d_\lambda d_\mu d_\nu\ldots).$$

Art. 88. Let monomial symmetric functions of the fictitious relation

$$x^r-D_1x^{r-1}+D_2x^{r-2}-\ldots = 0$$

be denoted by a partition in brackets with subscript D, thus,

$$(\quad)_D.$$

Then I have shown, in a previous paper,

$$d_1 = D_1 = (1)_D,$$
$$d_2 = D_1^2-2D_2 = (2)_D,$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$d_s = \cdot\quad\cdot\quad\cdot = (s)_D,$$

and, in general,

$$\frac{1}{\pi_1!\,\pi_2!\ldots}(d_{p_1}^{\pi_1}d_{p_2}^{\pi_2}\ldots) = (p_1^{\pi_1}p_2^{\pi_2}\ldots)_D.$$

Art. 89. Every symmetric function identity has corresponding to it a relation between the operators; thus corresponding to the set

$$(1)^2 = (2)+2(1^2),$$
$$(1)^3 = (3)+3(21)+6(1^3),$$
$$(1)^4 = (4)+4(31)+6(2^2)+12(21^2)+24(1^4),$$
$$\cdots\cdots\cdots\cdots$$

we have the set

$$(d_1)^2 = (d_1^{\,2})+d_2 = 2\,(1^2)_{\rm D}+(2)_{\rm D},$$
$$(d_1)^3 = (d_1^{\,3})+3\,(d_1d_2)+d_3 = 6\,(1^3)_{\rm D}+3\,(21)_{\rm D}+(3)_{\rm D},$$
$$(d_1)^4 = (d_1^{\,4})+6\,(d_1^{\,2}d_2)+3\,(d_2^{\,2})+4\,(d_3d_1)+(d_4),$$
$$= (24)\,(1^4)_{\rm D}+12\,(21^2)_{\rm D}+6\,(2^2)_{\rm D}+4\,(31)_{\rm D}+(4)_{\rm D},$$

and so on.

Art. 90. Also, corresponding to the set

$$2a_2 = s_1^{\,2}-s_2,$$
$$6a_3 = s_1^{\,3}-3s_1s_2+2s_3,$$
$$24a_4 = s_1^{\,4}-6s_1^{\,2}s_2+3s_2^{\,2}+8s_1s_3-6s_4,$$

&c., we have the set

$$2{\rm D}_2 = (d_1^{\,2}) = (d_1)^2-d_2,$$
$$6{\rm D}_3 = (d_1^{\,3}) = (d_1)^3-3\,(d_1)\,(d_2)+2d_3,$$
$$24{\rm D}_4 = (d_1^{\,4}) = (d_1)^4-6\,(d_1)^2\,(d_2)+3\,(d_2)^2+8\,(d_1)\,(d_3)-6\,(d_4),$$

and so on.

Art. 91. For the special operand

$$\frac{1}{1-{\rm A}}$$

these operator relations assume a special simple form which is of great importance in the theory of the generating function.

For
$$d_s{\rm A} = (-)^{s-1}b^{s-1}\,(1-b{\rm A}) = (-)^{s-1}b^{s-1}d_1{\rm A}\,;$$

or, *quâ* the above operand,
$$d_s \equiv (-)^{s-1}b^{s-1}d_1\,;$$

and thence, from a set of relations given above,

$$2\,!\,{\rm D}_2 \equiv {\rm D}_1\,({\rm D}_1+b),$$
$$3\,!\,{\rm D}_3 \equiv {\rm D}_1\,({\rm D}_1+b)\,({\rm D}_1+2b),$$
$$\cdots\cdots\cdots\cdots$$
$$s\,!\,{\rm D}_s \equiv {\rm D}_1\,({\rm D}_1+b)\ldots\{{\rm D}_1+(s-1)\,b\},$$
$$s\,!\,{\rm D}_s \equiv t\,!\,{\rm D}_t\,({\rm D}_1+tb)\,\{{\rm D}_1+(t+1)\,b\}\ldots\{{\rm D}_1+(s-1)\,b\}.$$

Art. 92. By means of these we can now arrive at a most important series of relations.

For

$$(p)_D = d_p = (-b)^{p-1} D_1 ;$$

$$\begin{aligned}(pq)_D &= (d_p d_q) = (d_p)(d_q) - d_{p+q},\\ &\equiv (-)^{p+q-2} b^{p+q-2} D_1^2 + (-)^{p+q-2} b^{p+q-1} D_1,\\ &\equiv (-b)^{p+q-2} 2!\, D_2 ;\end{aligned}$$

$$\begin{aligned}(pqr)_D &= (d_p d_q d_r)\\ &= (d_p)(d_q)(d_r) - (d_{p+q})(d_r) - (d_{p+r})(d_q)\\ &\quad - (d_{q+r})(d_p) + 2\, d_{p+q+r},\\ &\equiv \quad (-b)^{p+q+r-3} D_1^3,\\ &\quad -3(-b)^{p+q+r-2} D_1^2,\\ &\quad +2(-b)^{p+q+r-1} D_1,\\ &\equiv (-b)^{p+q+r-3} 3!\, D_3 ;\end{aligned}$$

and generally

$$(p_1 p_2 \ldots p_s)_D \equiv (-b)^{\Sigma p - s} s!\, D_s ;$$

and more generally

$$(p_1^{\pi_1} p_2^{\pi_2} \ldots)_D \equiv (-b)^{\Sigma \pi p - \Sigma \pi} \frac{(\Sigma \pi)!}{\pi_1!\, \pi_2! \ldots} D_{\Sigma \pi} ;$$

or, if $\Sigma \pi p = n$, $\Sigma \pi = i$,

$$(p_1^{\pi_1} p_2^{\pi_2} \ldots)_D \equiv (-b)^{n-i} \frac{i!}{\pi_1!\, \pi_2! \ldots} D_i .$$

Art. 93. From the relations

$$s!\, D_s = D_1 (D_1 + b) \ldots \{D_1 + (s-1) b\} ;$$

we find the set

$$\begin{aligned} D_1^2 &= 2D_2 - bD_1,\\ D_1^3 &= 6D_3 - 6bD_2 + b^2 D_1,\\ D_1^4 &= 24D_4 - 36bD_3 + 14b^2 D_2 - b^3 D_1,\\ &\ldots\ldots\ldots\ldots\end{aligned}$$

Art. 94. And also the set

$$\binom{s}{1} D_s = D_{s-1} \left\{ D_1 + \binom{s-1}{1} b \right\},$$

$$\binom{s}{2} D_s = D_{s-2} \left\{ D_2 + \binom{s-2}{1} bD_1 + \binom{s-1}{2} b^2 \right\},$$

$$\binom{s}{3} D_s = D_{s-3} \left\{ D_3 + \binom{s-3}{1} bD_2 + \binom{s-2}{2} b^2 D_1 + \binom{s-1}{3} b^3 \right\},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\binom{s}{t} D_s = D_{s-t} \left\{ D_t + \binom{s-t}{1} bD_{t-1} + \ldots + \binom{s-1}{t} b^t \right\},$$

$$sD_s = D_1 \{ D_{s-1} + bD_{s-2} + b^2 D_{s-3} + \ldots + b^{s-1} \}.$$

The Expressibility of D_s.

Art. 95. The fundamental relation

$$s!D_s = D_1(D_1+b)(D_1+2b)\ldots\{D_1+(s-1)b\},$$

exhibits D_s in terms of powers of D_1.

It is clear, *à priori*, that D_s is expressible in terms of D_1 and powers of D_2, *e.g.*,

$$\binom{3}{1}D_3 = D_2(D_1+2b),$$

$$\binom{4}{2}D_4 = D_2(D_2+2bD_1+3b^2),$$

$$\binom{3}{1}\binom{5}{3}D_5 = D_2(D_2D_1+8bD_2+9b^2D_1+12b^3),$$

$$\binom{4}{2}\binom{6}{4}D_6 = D_2(D_2^2+6bD_2D_1+29b^2D_2+24b^3D_1+30b^4), \text{ and so on,}$$

where notice, as a verification, that the sum of the numerical coefficients is the same on the two sides.

In every case D_2 appears as a factor.

In general the operator products, which appear on the right, are factors of

$$D_2^kD_1,$$

which contain the factor D_2, every weight of operator product being represented *once, and once only*, from the weight 2 up to the weight of the single operator on the left-hand side.

It is important to remark that

$$(2^k1)$$

is a perfect partition* of the number

$$2k+1;$$

because every lower number can be composed in exactly one way by the parts of the partition.

Art. 96. It will now appear that there exists an expression for

$$D_s$$

corresponding to every perfect partition that can be constructed.

The general expression of a perfect partition is

$$\ldots\{(1+\alpha)(1+\beta)(1+\gamma)\}^\delta\{(1+\alpha)(1+\beta)\}^\gamma(1+\alpha)^\beta1^\alpha;$$

where $\alpha, \beta, \gamma, \delta, \ldots$ are any positive integers, zero excluded.

The perfect partition

$$(2^k1)$$

is the particular case

$$\alpha = 1, \beta = k, \gamma = \delta = \ldots = 0.$$

* 'Messenger,' 1890, p. 103.

In every case, if σ be the highest figure in the perfect partition, D_σ is a factor of the expression for D_s, *e.g.*, taking the perfect partition

$$3^k1^2,$$

we have

$$40D_6 = 2D_3^2+3bD_3D_1^2+15b^2D_3D_1+20b^3D_3.$$

I do not interrupt the investigation by stopping to prove the theory of expressibility depending upon perfect partitions; its truth is intuitive.

Art. 97. It is necessary to labour the subject of the operator relations, *quâ* the special operands, because the whole theory of the numbers N_m is involved.

Art. 98. Perhaps the most interesting of the operator relations are those which do not involve b (or λ).

Recalling the relation of Art. 92, viz.,

$$(p_1^{\pi_1}p_2^{\pi_2}\ldots)_D = (-b)^{n-i}\frac{i!}{\pi_1!\pi_2!\ldots}D_i,$$

where

$$\Sigma\pi p = n, \ \Sigma\pi = i,$$

we may also write

$$(q_1^{\chi_1}q_2^{\chi_2}\ldots)_D = (-b)^{\nu-j}\frac{j!}{\chi_1!\chi_2!\ldots}D_j,$$

where

$$\Sigma\chi q = \nu, \ \Sigma\chi = j,$$

and, if

$$n-i = \nu-j,$$

we may eliminate b, obtaining

$$\frac{j!}{\chi_1!\chi_2!\ldots}D_j(p_1^{\pi_1}p_2^{\pi_2}\ldots)_D = \frac{i!}{\pi_1!\pi_2!\ldots}D_i(q_1^{\chi_1}q_2^{\chi_2}\ldots)_D.$$

Art. 99. The simplest formula thence obtained is found by putting

$$(p_1^{\pi_1}p_2^{\pi_2}\ldots) = (2^2),$$

$$(q_1^{\chi_1}q_2^{\chi_2}\ldots) = (31);$$

and this leads at once to

$$d_1d_3-d_2^2 = 0,$$

or

$$D_2D_1^2-4D_2^2+3D_3D_1 = 0,$$

which also results by elimination of b from

$$2D_2 = D_1(D_1+b),$$

$$6D_3 = D_1(D_1+b)(D_1+2b).$$

Art. 100. To obtain some more relations in a simple manner, I write

$$(s)_D = (-b)^{s-1}D_1,$$

$$(t)_D = (-b)^{t-1}D_1,$$

$$(u)_D = (-b)^{u-1}D_1,$$

$$(t+u)_D = (-b)^{t+u-1}D_1,$$

$$(u+s)_D = (-b)^{u+s-1}D_1,$$

$$(s+t)_D = (-b)^{s+t-1}D_1;$$

and then

$$(s+t)_D\,(u)_D = (s)_D\,(t+u)_D = (t)_D\,(u+s)_D\,;$$

or, as these relations may be written,

$$d_{s+t}\,d_u = d_s\,d_{t+u} = d_t\,d_{u+s}\,;$$

or

$$(s+t,\ u)_D = (s,\ t+u)_D = (t,\ u+s)_D,$$

with the usual multiplier (*viz.*, 2), if either

$$s+t = u, \quad \text{or} \quad t+u = s, \quad \text{or} \quad u+s = t.$$

Art. 101. We are led to the series

$$(31)_D = 2\,(2^2)_D,$$
$$(41)_D = (32)_D,$$
$$(51)_D = (42)_D = 2\,(3^2)_D,$$
$$(61)_D = (52)_D = (43)_D,$$
$$(71)_D = (62)_D = (53)_D = 2\,(4^2)_D, \ \&c.,$$

and generally if

$$(p_1^{\pi_1}p_2^{\pi_2}\ldots), \quad (q_1^{\chi_1}q_2^{\chi_2}\ldots)$$

be functions of the same weight and degree, viz.,

$$\Sigma\pi p = \Sigma\chi q\,;\ \Sigma\pi = \Sigma\chi = i,$$
$$\pi_1!\,\pi_2!\ldots(p_1^{\pi_1}p_2^{\pi_2}\ldots)_D = \chi_1!\,\chi_2!\ldots(q_1^{\chi_1}q_2^{\chi_2}\ldots)_D.$$

Application of the Foregoing to the Generating Function.

Art. 102. It has been established that

$$\frac{1}{1-a_1+(1-\lambda)\,a_2-(1-\lambda)^2a_3+\ldots}$$
$$= \Sigma N_{m,\,p_1p_2\ldots p_k}\lambda^{m-1}\,(p_1p_2\ldots p_k).$$

We will first of all examine the result of the equivalence of operators

$$2!D_2 = D_1^{\,2}+(1-\lambda)\,D_1$$

(see Art. 93 *quâ* the operand on the right-hand side). Write the operand

$$\Sigma N_{m,\,1^{c_1}2^{c_2}3^{c_3}\ldots}\lambda^{m-1}\,(1^{c_1}2^{c_2}3^{c_3}\ldots)\,;$$

then

$$2!N_{m,\,1^{c_1}2^{c_2}+1\,3^{c_3}\ldots}$$
$$= N_{m,\,1^{c_1}+2\,2^{c_2}3^{c_3}\ldots}+(N_m-N_{m-1})_{1^{c_1}+1\,2^{c_2}3^{c_3}\ldots},$$

e.g., put

$$c_1 = 0, \quad c_2 = 2, \quad c_3 = c_4 = \ldots = 0,$$
$$m = 3,$$
$$2N_{3,\,2^3} = N_{3,\,1^22^2}+(N_3-N_2)_{12^2}\,;$$

verified (from the tables) by

$$2.48 = 93+(15-12).$$

Art. 103. Again, in the same formula, put

$$c_1 = n-2, \quad c_2 = 0, \quad c_3 = c_4 = \ldots = 0,$$

we find

$$2N_{m,\, 21^{n-2}} = N_{m,\, 1^n} + (N_m - N_{m-1})_{1^{n-1}}.$$

We obtain, from this, a useful result by writing

$$n-m+1 \text{ for } m,$$

for then

$$2N_{n-m+1,\, 21^{n-2}} = N_{n-m+1,\, 1^n} + (N_{n-m+1} - N_{n-m})_{1^{n-1}}.$$

Art. 104. Observe that

$$N_{m,\, 1^n} = N_{n-m+1,\, 1^n},$$
$$N_{m,\, 1^{n-1}} = N_{n-m,\, 1^{n-1}},$$
$$N_{m-1,\, 1^{n-1}} = N_{n-m+1,\, 1^{n-1}};$$

so that by addition and subtraction we obtain

$$N_{m,\, 21^{n-2}} + N_{n-m+1,\, 21^{n-2}} = N_{m,\, 1^n};$$
$$N_{m,\, 21^{n-2}} - N_{n-m+1,\, 21^{n-2}} = N_{m,\, 1^{n-1}} - N_{n-m+1,\, 1^{n-1}};$$

or, as we may conveniently write these relations,

$$(N_m + N_{n-m+1})_{21^{n-2}} = N_{m,\, 1^n};$$
$$(N_m - N_{n-m+1})_{21^{n-2}} = (N_m - N_{n-m+1})_{1^{n-1}};$$
$$= (N_m - N_{m-1})_{1^{n-1}}.$$

These are the relations connecting N_m and N_{n-m+1} *quâ* the subscript 21^{n-2} analogous to those connecting the same symbols *quâ* the subscript 1^n.

Art. 105. From any operator relation we can immediately derive a relation between the numbers N_m by substituting for

$$b^\sigma D_s D_t \ldots$$

the expressions

$$\left\{N_m - \binom{\sigma}{1} N_{m-1} + \binom{\sigma}{2} N_{m-2} - \ldots\right\}_{1^{c_1}2^{c_2}\ldots s^{c_s+1}t^{c_t+1}\ldots};$$

and this it is convenient to denote by

$$N^{(\sigma)}{}_{m,\, 1^{c_1}2^{c_2}\ldots s^{c_s+1}t^{c_t+1}\ldots}.$$

Art. 106. Thus, corresponding to the operator relation

$$6D_3 = D_1(D_1+b)(D_1+2b) = D_1{}^3 + 3bD_1{}^2 + 2b^2D_1,$$

we obtain

$$6N_{m,\, 1^{c_1}2^{c_2}3^{c_3+1}\ldots} = N_{m,\, 1^{c_1+3}2^{c_2}3^{c_3}\ldots} + 3N^{(1)}{}_{m,\, 1^{c_1+2}2^{c_2}3^{c_3}\ldots} + 2N^{(2)}{}_{m,\, 1^{c_1+1}2^{c_2}3^{c_3}\ldots}.$$

As a particular case put

$$c_1 = n-3, \quad c_2 = c_3 = c_4 = \ldots = 0,$$

so that

$$6N_{m,\, 31^{n-3}} = N_{m,\, 1^n} + 3\,(N_m - N_{m-1})_{1^{n-1}} + 2\,(N_m - 2N_{m-1} + N_{m-2})_{1^{n-2}};$$

and thence

$$3(N_m+N_{n-m+1})_{31^{n-3}} = N_{m,1^n}+2(N_m-2N_{m-1}+N_{m-2})_{1^{n-2}};$$
$$(N_m-N_{n-m+1})_{31^{n-3}} = (N_m-N_{m-1})_{1^{n-1}}.$$

For $n = 6$, these relations can be verified by the tables for all values of m.

Art. 107. Similarly the theorems derived from

$$s!\,D_s = D_1(D_1+b)(D_1+2b)\ldots\{D_1+(s-1)b\}$$

can be at once written down.

[It is worth noting that this operator relation can, by putting $D_1 = b\Delta_1$, be written

$$D_s = b^s\binom{\Delta_1+s-1}{s}\Bigg].$$

If

$$s_1!\,s_2!\ldots D_{s_1}D_{s_2}\ldots = D_1^{\Sigma s}+g_1D_1^{\Sigma s-1}+g_2D_1^{\Sigma s-2}+\ldots,$$

we can write down the corresponding relation between the numbers N_m.

It will be found that

$$(N_m+N_{n-m+1})_{s1^{n-s}}$$

is a linear function of

$$N_{m,1^n},\ N^{(2)}_{m,1^{n-2}},\ N^{(4)}_{m,1^{n-4}}\ldots;$$

and

$$(N_m-N_{n-m+1})_{s1^{n-s}}$$

a linear function of

$$N^{(1)}_{m,1^{n-1}},\ N^{(3)}_{m,1^{n-3}},\ N^{(5)}_{m,1^{n-5}}\ldots;$$

and that the same obtains when instead of

$$s1^{n-s}$$

we take

$$s_1s_2\ldots1^{n-\Sigma s}.$$

Art. 108. From the operator relation

$$(p_1!)^{\pi_1}(p_2!)^{\pi_2}\ldots D_{p_1}^{\pi_1}D_{p_2}^{\pi_2}\ldots = \prod_{p,\pi}[D_1(D_1+b)\ldots\{D_1+(p_1-1)b\}]^{\pi_1}$$

we find

$$D_{p_1}^{\pi_1}D_{p_2}^{\pi_2}\ldots = b^n\binom{\Delta_1+p_1-1}{p_1}^{\pi_1}\binom{\Delta_1+p_2-1}{p_2}^{\pi_2}\ldots,$$
$$= b^n(u_0\Delta_1^n+u_1\Delta_1^{n-1}+u_2\Delta_1^{n-2}+\ldots);$$

where u_0, u_1, u_2, ... are numerical coefficients that may be determined.

Thence is derived the relation

$$N_{m,p_1^{\pi_1}p_2^{\pi_2}\ldots}$$
$$= u_0N_{m,1^n}+u_1N^{(1)}_{m,1^{n-1}}+u_2N^{(2)}_{m,1^{n-2}}+\ldots;$$

giving a hint to put

$$N^{(t)}_{m,1^{n-t}} = N_m^{n-t} \text{ symbolically},$$

and then

$$N_{m,p_1^{\pi_1}p_2^{\pi_2}\ldots} = \binom{N_m+p_1-1}{p_1}^{\pi_1}\binom{N_m+p_2-1}{p_2}^{\pi_2}\ldots \text{ symbolically.}$$

Art. 109. It must now be remarked that, since

$$N_m^{n-t} = \left\{N_m-\binom{t}{1}N_{m-1}+\binom{t}{2}N_{m-2}\ldots\right\}_{1^{n-t}},$$

we obtain

$$N^{n-t}_{n-m+1} = \left\{N_{n-m+1} - \binom{t}{1} N_{n-m} + \binom{t}{2} N_{n-m-1} - \ldots\right\}_{1^{n-t}};$$

and since

$$N_{s, 1^{n-t}} = N_{n-s-t+1, 1^{n-t}}, \quad N^{n-t}_{n-m+1} = (-)^t N_m{}^{n-t};$$

and

$$N_{n-m+1},\ p_1{}^{\pi_1} p_2{}^{\pi_2}\ldots$$

$$= u_0 N_{m, 1^n} - u_1 N^{(1)}{}_{m, 1^{n-1}} + u_2 N^{(2)}{}_{m, 1^{n-2}} - \ldots$$

$$= u_0 N_m{}^n - u_1 N_m{}^{n-1} + u_2 N_m{}^{n-2} - \ldots$$

$$= \binom{N_m}{p_1}^{\pi_1} \binom{N_m}{p_2}^{\pi_2} \ldots;$$

we obtain

$$N_{m, p_1{}^{\pi_1} p_2{}^{\pi_2}\ldots}$$

$$= \binom{N_{n-m+1}}{p_1}^{\pi_1} \binom{N_{n-m+1}}{p_2}^{\pi_2} \ldots.$$

Art. 110. We have two alternative expressions for

$$N_{m, p_1{}^{\pi_1} p_2{}^{\pi_2}\ldots}$$

in terms of numbers

$$N_{\mu, 1^\sigma}.$$

I verify them in the case

$$p_1{}^{\pi_1} p_2{}^{\pi_2}\ldots = 2^3;$$

(i.)

$$N_{m, 2^3} = \tfrac{1}{8}(N_m{}^2 + N_m)^3,$$

or

$$8N_{m, 2^3} = N_m{}^6 + 3N_m{}^5 + 3N_m{}^4 + N_m{}^3,$$

$$= N_{m, 1^6}$$
$$+ 3(N_m - N_{m-1})_{1^5}$$
$$+ 3(N_m - 2N_{m-1} + N_{m-2})_{1^4}$$
$$+ (N_m - 3N_{m-1} + 3N_{m-2} - N_{m-3})_{1^3},$$

agreeing with, for $m = 3$,

$$8.48 = 302 + 3(66 - 26) + 3(11 - 2.11 + 1) + (1 - 3.4 + 3.1).$$

(ii.)

$$N_{m, 2^3} = \tfrac{1}{8}(N^2{}_{7-m} - N_{7-m})^3,$$

or

$$8N_{m, 2^3} = N^6{}_{7-m} - 3N^5{}_{7-m} + 3N^4{}_{7-m} - N^3{}_{7-m},$$

and, for $m = 3$,

$$8N_{3, 2^3} = N_4{}^6 - 3N_4{}^5 + 3N_4{}^4 - N_4{}^3,$$

$$= N_{4, 1^6}$$
$$- 3(N_4 - N_3)_{1^5}$$
$$+ 3(N_4 - 2N_3 + N_2)_{1^4}$$
$$- (N_4 - 3N_3 + 3N_2 - N_1)_{1^3};$$

agreeing with

$$8.48 = 302 - 3(26 - 66) + 3(1 - 2.11 + 11) - (-3.1 + 3.4 - 1).$$

Art. 111. We have seen that, in general, we have two expressions for

$$N_{m,\, p_1^{\pi_1} p_2^{\pi_2} \ldots},$$

but, since

$$N_{m,\, p^{\pi}} = N_{n-m-p+2,\, p^{\pi}},$$

we have four expressions for

$$N_{m,\, p^{\pi}},$$

viz.,

$$\binom{N_m + p - 1}{p}^{\pi},$$

$$\binom{N_{n-m+1}}{p}^{\pi},$$

$$\binom{N_{n-m-p+2} + p - 1}{p}^{\pi},$$

$$\binom{N_{m+p-1}}{p}^{\pi}.$$

Art. 112. It is clear that the operator relations afford unlimited scope for obtaining theorems connecting the numbers

$$N_{m,\, p_1^{\pi_1} p_2^{\pi_2} \ldots}.$$

Relations, so far utilised, have involved the operator

$$D_1,$$

but it is easy to construct them so as not to contain D_1 and generally so as not to contain D_s, where s is less than a given integer.

E.g., from the symmetric function relation

$$(1^2)(1^2) = (2^2) + 2(21^2) + 6(1^4),$$

we find

$$D_2^2 = b^2 D_2 - 6bD_3 + 6D_4;$$

and generally the relation

$$(p^2)(q^2) = (\overline{p+q}^2) + (p+q,\, pq) + (p^2 q^2)$$

leads to

$$(-b)^{2p+2q-4} D_2^2 = (-b)^{2p+2q-2} D_2 + 6(-b)^{2p+2q-3} D_3 + 6(-b)^{2p+2q-4} D_4;$$

or, throwing out the factor

$$b^{2p+2q-4},$$

$$D_2^2 = b^2 D_2 - 6bD_3 + 6D_4,$$

the same relation as before.

Moreover, the relation

$$(pq)(rs) = (p+r,\, q+s) + (p+s,\, q+r) + (p+r,\, qs) + (q+r,\, ps)$$
$$+ (p+s,\, qr) + (q+s,\, pr) + (pqrs)$$

leads, after throwing out a power of b, to precisely the same relation.

Art. 113. This remarkable circumstance greatly limits the number of operator relations obtainable. It should be observed that *any* operator relation may be multiplied throughout by any power of b and may be then used to obtain relations between the numbers

$$N_{m,\, p_1^{\pi_1} p_2^{\pi_2} \ldots},$$

but no essentially new relations are thus obtainable; for take a simple case

$$2D_2 = D_1^2 + bD_1,$$

leading to

$$2N_{m, 1^{c_1}2^{c_2+1}\dots} = N_{m, 1^{c_1+2}2^{c_2}\dots} + (N_m - N_{m-1})_{1^{c_1+1}2^{c_2}\dots},$$

true for all values of m.

If we take

$$2bD_2 = bD_1^2 + b^2D_1,$$

we are led to

$$2(N_m - N_{m-1})_{1^{c_1}2^{c_2+1}\dots} = (N_m - N_{m-1})_{1^{c_1+2}2^{c_2}\dots} + (N_m - 2N_{m-1} + N_{m-2})_{1^{c_1+1}2^{c_2}\dots},$$

and if the former relation be written

$$f(m) = 0,$$

the latter is merely

$$f(m) - f(m-1) = 0;$$

and further multiplication by b leads to the series of which the general term is

$$f(m) - \binom{\sigma}{1} f(m-1) + \binom{\sigma}{2} f(m-2) \dots;$$

so that no new information is obtained.

Art. 114. The operator relation of the form D_sD_t = a linear function of

$$b^tD_s,\ b^{t-1}D_{s+1},\ b^{t-2}D_{s+2},\dots$$

is not difficult to obtain.

I find that

$$(-)^tD_sD_t = \binom{s}{t}\binom{t}{0} b^tD_s - \binom{s+1}{t}\binom{1}{t} b^{t-1}D_{s+1}$$
$$+ \binom{s+2}{t}\binom{t}{2} b^{t-2}D_{s+2} - \dots;$$

and thence the formula for

$$D_sD_tD_u$$

follows by taking D_u as the operand on each side and then reducing the products

$$D_sD_u,\ D_{s+1}D_u,\ D_{s+2}D_u,\dots$$

by the formula for D_sD_t.

I find that

$$(-)^{t+u}D_sD_tD_u$$
$$= \binom{s}{u} \binom{t}{0}\binom{s}{t}\binom{u}{0} b^{t+u}D_s$$
$$- \binom{s+1}{u}\left\{\binom{t}{0}\binom{s}{t}\binom{u}{1} + \binom{t}{1}\binom{s+1}{t}\binom{u}{0}\right\} b^{t+u-1}D_{s+1}$$
$$+ \binom{s+2}{u}\left\{\binom{t}{0}\binom{s}{t}\binom{u}{2} + \binom{t}{1}\binom{s+1}{t}\binom{u}{1} + \binom{t}{2}\binom{s+2}{t}\binom{u}{0}\right\} b^{t+u-2}D_{s+2}$$
$$- \dots;$$

and, generally, the product of any number of operators is expressible in the required linear form.

Art. 115. With the object of connecting this theory of the numbers N_m with that of the numbers N $(abc\ldots)$, the generating function

$$\frac{1}{1-\alpha_1+(1-\lambda)\,\alpha_2-(1-\lambda)^2\alpha_3+\ldots}$$

will now be expanded in ascending powers of λ, the coefficients of λ being functions of the homogeneous product sums

$$h_1,\ h_2,\ h_3,\ \ldots.$$

The point of departure is the elementary formula

$$\frac{1}{1-\alpha_1+\alpha_2-\alpha_3+\ldots} = 1+h_1+h_2+h_3+\ldots.$$

Remarking that

$$1-\alpha_1+\alpha_2-\alpha_3+\ldots = (1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\ldots,$$

I write

$$(1-\lambda)\,\alpha_s \text{ for } \alpha_s,$$

equivalent to writing

$$(1-\lambda)^s\alpha_s \text{ for } \alpha_s,$$

and

$$(1-\lambda)^s h_s \text{ for } h_s;$$

then

$$\frac{1}{1-(1-\lambda)\,\alpha_1+(1-\lambda)^2\alpha_2-(1-\lambda)^3\alpha_3+\ldots} = 1+(1-\lambda)\,h_1+(1-\lambda)^2h_2+(1-\lambda)^3h_3+\ldots$$

$$= u \text{ suppose};$$

and, as before, write

$$\alpha_1-(1-\lambda)\,\alpha_2+(1-\lambda)^2\alpha_3-\ldots = \mathrm{A};$$

so that

$$u = \frac{1}{1-(1-\lambda)\,\mathrm{A}};$$

whence, solving for A,

$$\mathrm{A} = \frac{1}{1-\lambda}\,\frac{u-1}{u},$$

and

$$\frac{1}{1-\mathrm{A}} = (1-\lambda)\,\frac{u}{1-\lambda u};$$

where $\frac{1}{1-\mathrm{A}}$ is the generating function under consideration.

Write

$$\mathrm{H}_1 = h_1+h_2+h_3+h_4+\ldots$$

$$\mathrm{H}_2 = h_2+2h_3+3h_4+4h_5+\ldots$$

$$\mathrm{H}_3 = h_3+3h_4+6h_5+10h_6+\ldots$$

$$\mathrm{H}_4 = h_4+4h_5+10h_6+20h_7+\ldots$$

. . . .

so that
$$u = 1+(1-\lambda)\,h_1+(1-\lambda)^2 h_2+(1-\lambda)^3 h_3+\ldots$$
$$= 1+H_1-\lambda\,(H_1+H_2)+\lambda^2\,(H_2+H_3)+\ldots+(-)^s(H_s+H_{s+1})+\ldots$$
$$= 1+(1-\lambda)\,H_1-\lambda\,(1-\lambda)\,H_2+\lambda^2\,(1-\lambda)\,H_3-\ldots;$$

therefore
$$\frac{1-\lambda u}{1-\lambda} = 1-\lambda H_1+\lambda^2 H_2-\lambda^3 H_3+\ldots$$

and thence
$$\frac{1}{1-A} = 1+\frac{H_1-\lambda H_2+\lambda^2 H_3-\ldots}{1-\lambda H_1+\lambda^2 H_2-\lambda^3 H_3+\ldots}.$$

Now let functions $A_1,\ A_2,\ A_3,\ A_4,\ \ldots$

be connected with $H_1,\ H_2,\ H_3,\ H_4,\ \ldots$

in the same way that $a_1,\ a_2,\ a_3,\ a_4,\ \ldots$

are connected with $h_1,\ h_2,\ h_3,\ h_4,\ \ldots$

so that
$$A_1 = H_1,$$
$$A_2 = H_1^2-H_2,$$
$$A_3 = H_1^3-2H_1H_2+H_3,$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

then
$$\frac{1}{1-\lambda H_1+\lambda^2 H_2-\lambda^3 H_3+\ldots} = 1+\lambda A_1+\lambda^2 A_2+\lambda^3 A_3+\ldots,$$

and
$$\frac{1}{1-A} = 1+(H_1-\lambda H_2+\lambda^2 H_3-\ldots)\,(1+\lambda A_1+\lambda^2 A_2+\lambda^3 A_3+\ldots).$$

On the dexter the co-factor of λ^s is
$$H_1A_s-H_2A_{s-1}+H_3A_{s-2}-\ldots+(-)^s H_{s+1},$$
which has the value
$$A_{s+1},$$
since
$$a_{s+1}-h_1a_s+h_2a_{s-1}-\ldots+(-)^{s+1}h_{s+1} = 0$$
is a well-known identity in the elementary theory of symmetric functions. Hence
$$\frac{1}{1-A} = 1+A_1+\lambda A_2+\lambda^2 A_3+\lambda^3 A_4+\ldots;$$
or, as we may write it,
$$\frac{1}{1-a_1+(1-\lambda)\,a_2-(1-\lambda)^2\,a_3+\ldots}$$
$$= 1+A_1+\lambda A_2+\lambda^2 A_3+\lambda^3 A_4+\ldots,$$
$$= 1+H_1+\lambda H_{11}+\lambda^2 H_{111}+\lambda^3 H_{1111}+\ldots,$$
$$= 1+H_1+\lambda\,(H_1^2-H_2)+\lambda^2\,(H_1^3-2H_1H_2+H_3)$$
$$+\lambda^3\,(H_1^4-3H_1^2H_2+H_2^2+2H_1H_3-H_4)+\ldots$$

Art. 116. The preceding pages show that the coefficient of

$$\lambda^{m-1}$$

in the expansion of

$$\frac{1}{1-a_1+(1-\lambda)a_2-(1-\lambda)^2a_3+\ldots}$$

is equal to

$$\Sigma N_{m,(pqr\ldots)},$$

the summation being for every partition

$$(pqr\ldots);$$

and this, from the theory of the numbers

$$N(p_1p_2\ldots p_m),$$

is equal to

$$\sum_{p_1}\sum_{p_2}\ldots\sum_{p_m} h_{p_1p_2\ldots p_m},$$

the summation being for all integer values of

$$p_1, p_2, \ldots p_m;$$

or, the same thing, for the compositions of all numbers into exactly m parts.

Hence

$$\Sigma h_{p_1} = A_1 = H_1,$$

$$\Sigma\Sigma h_{p_1p_2} = A_2 = H_{11} = H_1^2 - H_2,$$

$$\Sigma\Sigma\Sigma h_{p_1p_2p_3} = A_3 = H_{111} = H_1^3 - 2H_1H_2 + H_3,$$

. . . .

$$\Sigma\Sigma\ldots\Sigma h_{p_1p_2\ldots p_m} = A_m = H_1^m,$$

a remarkable result.

Art. 117. Since

$$\frac{1}{1-a_1+(1-\lambda)a_2-(1-\lambda)^2a_3+\ldots}$$

$$= 1+\Sigma h_{p_1}+\lambda\Sigma h_{p_1p_2}+\lambda^2\Sigma h_{p_1p_2p_3}+\ldots,$$

we find, putting $\lambda = 1$,

$$\frac{1}{1-a_1} = 1+\Sigma h_{p_1}+\Sigma h_{p_1p_2}+\Sigma h_{p_1p_2p_3}+\ldots,$$

and, since

$$D_1 = d_1 \equiv \partial_{a_1},$$

$$D_1^s(1+\Sigma h_{p_1}+\Sigma h_{p_1p_2}+\Sigma h_{p_1p_2p_3}+\ldots)$$

$$= s!\,(1+\Sigma h_{p_1}+\Sigma h_{p_1p_2}+\Sigma h_{p_1p_2p_3}+\ldots)^{s+1};$$

and thence, as an easy deduction,

$$D_s(1+\Sigma h_{p_1}+\Sigma h_{p_1p_2}+\Sigma h_{p_1p_2p_3}+\ldots)$$
$$=(1+\Sigma h_{p_1}+\Sigma h_{p_1p_2}+\Sigma h_{p_1p_2p_3})^{s+1}$$

$\left(\text{for observe that, for the operand } \frac{1}{1-\alpha_1},\ D_1{}^s \equiv s!\,D_s\right)$, and thence, by an easy step,

$$D_{p_1}{}^{\pi_1}D_{p_2}{}^{\pi_2}\ldots(1+\Sigma h_{p_1}+\Sigma h_{p_1p_2}+\Sigma h_{p_1p_2p_3}+\ldots)$$
$$=\frac{(\Sigma\pi p)!}{(p_1!)^{\pi_1}(p_2!)^{\pi_2}\ldots}(1+\Sigma h_{p_1}+\Sigma h_{p_1p_2}+\Sigma h_{p_1p_2p_3}+\ldots)^{\Sigma\pi p+1}.$$

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